Self-similar intermediate asymptotics for a degenerate parabolic filtration-absorption equation

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The equation $\partial_t u = u\partial^2_{xx}u - (c-1)(\partial_x u)^2$ is known in literature as a qualitative mathematical model of some biological phenomena. Here this equation is derived as a model of the groundwater flow in a water-absorbing fissurized porous rock; therefore, we refer to this equation as a filtration-absorption equation. A family of self-similar solutions to this equation is constructed. Numerical investigation of the evolution of non-self-similar solutions to the Cauchy problems having compactly supported initial conditions is performed. Numerical experiments indicate that the self-similar solutions obtained represent intermediate asymptotics of a wider class of solutions when the influence of details of the initial conditions disappears but the solution is still far from the ultimate state: identical zero. An open problem caused by the nonuniqueness of the solution of the Cauchy problem is discussed.

1. A Derivation of the Filtration-Absorption Equation

A permeable rock layer on a horizontal impermeable bed is considered. It is well known that the gently sloping groundwater flows in a purely porous medium with infiltration are described by the Boussinesq equation (see refs. 1–3):

$$m\partial_t h = \kappa\left(h\partial^2_{xx}h + (\partial_x h)^2\right) + q. \qquad [1]$$

Here h is the groundwater level, t is the time, x is the horizontal space coordinate along the impermeable bed, and q is the intensity of the groundwater inflow or outflow. Furthermore, $\kappa = \rho g k/\mu$ is a joint property of the pair-rock/fluid, assumed here to be constant, k is the rock permeability, m the rock porosity, g the gravity acceleration, and ρ and μ are fluid density and dynamic viscosity.

Assume now that when the fluid level is decreasing, some part of the fluid is being absorbed by rock, e.g., because of capillary imbibition to the micropores. Then, for a fixed fluid particle, the rate of fluid absorption q will be proportional to the *individual* time derivative of the fluid level dh/dt, so that

$$q = \alpha m\frac{dh}{dt} = \alpha m(\partial_t h + u_p\partial_x h). \qquad [2]$$

Here α is also a joint rock/fluid property that we also assume here to be constant, and u_p is the actual fluid velocity.

According to the Darcy law, the filtration velocity (fluid flux per unit area) is equal to

$$u_f = -\frac{\rho g k}{\mu}\partial_x h. \qquad [3]$$

The actual fluid velocity is different (see ref. 1) from the filtration velocity u_f. For a purely porous medium $u_p = u_f/m$. It is very important that if the rock is fissurized, i.e., contains a connected network of cracks, then

$$u_p = \frac{u_f}{m_1}, \qquad [4]$$

where m_1 ("*fissure porosity*") is much less than block porosity m: fluid is contained in pores but moves through cracks that are much wider than the pores but occupy much less space. Therefore,

$$q = \alpha m\left(\partial_t h - \frac{\rho g k}{\mu m_1}(\partial_x h)^2\right). \qquad [5]$$

Substitution of **5** to the water balance equation **1** gives the equation

$$m(1-\alpha)\partial_t h = \frac{\rho g k}{\mu}\left(h\partial^2_{xx}h + \left(1 - \alpha\frac{m}{m_1}\right)(\partial_x h)^2\right),$$

which can be reduced to the form

$$\partial_t h = \frac{\kappa}{m(1-\alpha)}(h\partial^2_{xx}h - (c-1)(\partial_x h)^2), \qquad [6]$$

where

$$c = \alpha\frac{m}{m_1}. \qquad [7]$$

Here there is a specially important point to be mentioned: if $m_1 = m$ (non-fissurized purely porous medium), then $c = \alpha$ and obviously c is always less than one because the absorption cannot exceed the available amount of fluid. However, if the rock is fissurized, m_1 can be substantially less than m, and $c = \alpha m/m_1$ can be substantially larger than one. Replacing x by $x/\sqrt{\kappa/m(1-\alpha)}$ and leaving previous notation x for transformed space coordinate, we reduce the basic equation to the canonic form

$$\partial_t h = h\partial^2_{xx}h - (c-1)(\partial_x h)^2, \qquad [8]$$

which will be investigated further.

Note that the proposed model has some common features with the Mirzadzhan–Zadeh model (4) of filtration of the gas-condensate mixture, but is not identical to this model which leads to a different basic equation.

2. Self-Similar Solutions

We look for self-similar solutions with shrinking support and with finite time to collapse (total annihilation): for sufficiently large c (in fact, see later, for $c > 3/2$)

$$h = A(t_0 - t)^\lambda F\left(\frac{x - x_0}{B(t_0 - t)^\mu}\right). \qquad [9]$$

Here x_0 is the point where the solution is collapsing at $t = t_0$. We assume the structure of the groundwater "dome" is symmetric, so that $F(\xi)$, $\xi = (x - x_0)/B(t_0 - t)^\mu$ is an even function. Here A, B, and λ are constants, but as we will see later, of a different nature. We will determine the function $F(\xi)$ in the interval $0 \le \xi \le 1$, so that $F(\xi) \equiv 0$ at $\xi \ge 1$. The quantity

$$x_f = B(t_0 - t)^\mu \qquad [10]$$

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is the contracting half-width of the groundwater dome. As is well known (5) for the degenerate parabolic differential equations of the type under consideration, the support of the solution remains compact if it is compact initially. Furthermore, the function F can be normalized arbitrarily, so we can assume $A = B^2\mu$. Substitute **9** into the basic equation **8**. Bearing in mind that the coefficients of the resulting equation for $F(\xi)$ cannot contain the time t explicitly, we obtain $\lambda = 2\mu - 1$, and the equation for $F(\xi)$ assumes the form:

$$F\frac{d^2F}{d\xi^2} - (c-1)\left(\frac{dF}{d\xi}\right)^2 - \xi\frac{dF}{d\xi} + \frac{2\mu-1}{\mu}F = 0. \quad [11]$$

We turn now to the boundary conditions. The first one is the condition of symmetry:

$$F'(0) = 0. \quad [12]$$

The second boundary condition follows from the continuity of the groundwater level at the free boundary $x = x_f$:

$$F(1) = 0. \quad [13]$$

The last boundary condition follows from the continuity of the groundwater flux at the free boundary $x = x_f$. The solution close to free boundary can be considered as quasi-steady: $h = h(\zeta)$, $\zeta = x - x_f$. We obtain from **8**:

$$-V\frac{dh}{d\zeta} = h\frac{d^2h}{d\zeta^2} - (c-1)\left(\frac{dh}{d\zeta}\right)^2, \quad [14]$$

where $V = dx_f/dt$.

For the solutions of Eq. **11** having the quantity $d(F^2)/d\xi$ equal to zero at $\xi = 1 - 0$, which is needed to have continuous flux at the free boundary, the quantity $hd^2h/d\zeta^2$ tends to zero at $x = x_f - 0$, so that at $x = x_f$

$$V = (c-1)\frac{dh}{d\zeta}. \quad [15]$$

Bearing in mind that for the self-similar solution **9**, $V = dx_f/dt = -B\mu(t_0 - t)^{\mu-1}$, and $(dh/d\zeta)_{\zeta=0} = (\partial_x h)_{x=x_f}$, we obtain the relation

$$B\mu(t_0 - t)^{\mu-1} = -B\mu(t_0 - t)^{\mu-1}(c-1)F'(1)$$

from which the third boundary condition follows, to be satisfied by the function F

$$F'(1) = -\frac{1}{c-1}. \quad [16]$$

So, a nonlinear eigenvalue problem is obtained. We have to find for the *second order* equation **11** in the interval $[0, 1]$ the solution satisfying *three* boundary conditions (**12**, **13**, **16**) and the eigenvalue μ.

We consider the special case when the function $F(\xi)$ has a maximum at $\xi = 0$. This allows one to search the solution in the form of an expansion

$$F(\xi) = a(1 - \xi^2) + \sum_{n=2}^{\infty} a_n(1 - \xi^2)^n,$$

so that the terms of the sum (except the first one) do not contribute to all three boundary conditions of the eigenvalue problem. The result is unexpectedly simple:

$$a = \frac{1}{2(c-1)}, \qquad \mu = \frac{c-1}{2c-3}, \qquad a_n = 0\ (n \geq 2), \quad [17]$$

so that, if $c > 3/2$, the solution to the nonlinear eigenvalue problem is obtained in the form

$$F = \frac{1}{2(c-1)}(1 - \xi^2), \qquad \mu = \frac{c-1}{2c-3} \quad [18]$$

and the self-similar solution under consideration is represented by the relation

$$h = \frac{1}{2(2c-3)}B^2(t_0 - t)^{\frac{1}{2c-3}}\left[1 - \frac{(x - x_0)^2}{B^2(t_0 - t)^{\frac{2(c-1)}{2c-3}}}\right]. \quad [19]$$

3. Investigation of the Self-Similar Solutions

In spite of its very simple form, the solution **19** is a typical self-similar solution of the second kind (see ref. 6). The exponent $\mu = (c-1)/(2c-3)$ cannot be obtained using some conservation laws, but only by solving a nonlinear eigenvalue problem, and the constants B and t_0, as well as x_0, are obtained by matching the self-similar solution with the solution to the Cauchy problem at the non-self-similar stage. To a certain extent, this problem is similar to the problem of the evolution of a turbulent burst (see ref. 6, section 10.2.4). The solution **19** has essentially different behavior in various intervals of the values of the absorption coefficient c:

$$0 < c < 1; \qquad 1 < c < \tfrac{3}{2}; \qquad \tfrac{3}{2} < c. \quad [20]$$

The form **19** is appropriate for the last interval $3/2 < c$ where the collapse time t_0 is finite. It is instructive to investigate the limiting behavior of the solution **19** at $c \to 3/2$ from above. Putting $c = 3/2 + \varepsilon$ ($\varepsilon > 0$ is a small parameter), we obtain $\mu = 1/4\varepsilon + \frac{1}{2}$, so that

$$(t_0 - t)^{\mu} = t_0^{\frac{1}{4\varepsilon}+\frac{1}{2}}\left(1 - \frac{t}{t_0}\right)^{\frac{1}{4\varepsilon}+\frac{1}{2}}, \quad (t_0 - t)^{2\mu-1} = t_0^{\frac{1}{2\varepsilon}}\left(1 - \frac{t}{t_0}\right)^{\frac{1}{2\varepsilon}},$$

and

$$x_f(t) = Bt_0^{\frac{1}{4\varepsilon}+\frac{1}{2}}\left(1 - \frac{t}{t_0}\right)^{\frac{1}{4\varepsilon}+\frac{1}{2}}, \quad h(x_0, t) = \frac{B^2}{4\varepsilon}t_0^{\frac{1}{2\varepsilon}+1}\frac{1}{t_0}. \quad [21]$$

Therefore, if at $\varepsilon \to 0$ the quantity $4\varepsilon t_0$ tends to a certain constant Θ, and the quantity $B^2\ t_0^{\frac{1}{2\varepsilon}+1}$ to another constant which we denote by $C^2\Theta$, the solution **19** tends to a finite limit:

$$h = C^2 e^{-2t/\Theta}\left[1 - \frac{(x - x_0)^2}{C^2\Theta e^{-2t/\Theta}}\right], \qquad x_f = C\sqrt{\Theta}e^{-t/\Theta}. \quad [22]$$

In the interval $1 < c < 3/2$, the exponent μ becomes negative, and it is convenient to replace μ by $-\mu$, and t_0 by $-t_0$. Solution **19** may be represented in a different form

$$h = \frac{1}{2(3-2c)}B^2(t_0 + t)^{-\frac{1}{3-2c}}\left[1 - \frac{(x - x_0)^2}{B^2(t_0 + t)^{-\frac{2(c-1)}{3-2c}}}\right], \quad [23]$$

so that $h(x_0, t) = h_{\max}(t)$ and x_f decay with time according to the power laws

$$h_{\max}(t) = \frac{1}{2(3-2c)}B^2(t_0 + t)^{-\frac{1}{3-2c}}, \qquad x_f(t) = B(t_0 + t)^{-\frac{c-1}{3-2c}}. \quad [24]$$

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The time of collapse is infinite and t_0 becomes simply an additive constant. In the limit $c \to 3/2$ from below, $c = 3/2 - \varepsilon$, $\varepsilon > 0$, we obtain $\mu = 1/4\varepsilon - 1/2$, and

$$h_{\max}(t) = \frac{1}{4\varepsilon} B^2 t_0^{-\frac{1}{2\varepsilon}+1}\left(1+\frac{t}{t_0}\right)^{-\frac{1}{2\varepsilon}+1} \cdot \frac{1}{t_0},$$

$$x_f(t) = B t_0^{-\frac{1}{4\varepsilon}+\frac{1}{2}}\left(1+\frac{t}{t_0}\right)^{-\frac{1}{4\varepsilon}+\frac{1}{2}}.$$

Assuming again that at $\varepsilon \to 0$ the quantity $4\varepsilon t_0$ tends to a constant Θ, and $B t_0^{-\frac{1}{4\varepsilon}+\frac{1}{2}}$ tends to another constant $C\sqrt{\Theta}$, we obtain the same limiting formula **22**.

In the interval $0 < c < 1$ (weak absorption), the compact support extends, not contracts, although more slowly than in the case of "porous medium equation" $c = 0$. In this special case $c = 0$ solution **19** is reduced to a known self-similar solution fo the first kind (refs. 7 and 8; see also refs. 6 and 9). The degenerate special case $c = 1$ was considered previously; the papers by J. R. King (10) and P. Rosenau (11) should be mentioned specially. It is instructive to compare the results obtained above with those obtained in the paper by B. Meerson *et al.* (12).

4. Nonuniqueness of Solutions of the Cauchy Problem

We consider solutions of Eq. **18** with initial condition

$$h(x, 0) = h_0(x) \qquad \text{for } x \in \mathbb{R}, \qquad \textbf{[25]}$$

where $h_0(x)$ is a continuous function that is positive in an interval $(x_L(0), x_R(0))$ and that vanishes elsewhere. Let $x_L(t)$ and $x_R(t)$ be two continuous functions for $t \geq 0$ such that $x_L(t)$ is nondecreasing, $x_R(t)$ is nonincreasing, and $x_L(t) \leq x_R(t)$ for $t \geq 0$. It is known that if $c \geq 1$ for any such pair $x_L(t)$ and $x_R(t)$, there exists a solution $h(x, t)$ of the Cauchy problem (**8**, **25**) such that $h(x, t)$ is positive if $x_L(t) < x < x_R(t)$, $t \geq 0$ and $h(x, t)$ vanishes elsewhere. For the proof, the definition of solution, and further references we refer to (13).

Of special interest is the choice of steady interfaces: $x_L(t) = x_L(0)$ and $x_R(t) = x_R(0)$ for all $t \geq 0$. The corresponding solution is larger than any other solution, and in ref. 14 a numerical scheme has been introduced that leads to this unique solution. From the modeling point of view it is interesting to observe that this solution can be obtained from the following limiting procedure: replace $h_0(x)$ by $h_0(x) + \varepsilon$ ($\varepsilon > 0$), solve problem **8**, **25**, and let $\varepsilon \to 0$.

Angenent (15) has constructed a solution of **8**, **25** if $h_0(x)$ has nonzero slope at $x_L(0)$ and $x_R(0)$ (for technical reasons the construction is local in time), which is unique in the class of solutions that can be expanded in a Taylor series of suffcieintly high degree near the interfaces:

$$h(x, t) = \sum_{k=0}^{N} c_k(t)(x - x_L(t))^k + o((x - x_L(t))^N) \qquad \text{as } x \to x_L(t);$$

a similar expression holds at the right interface $x = x_R(t)$. Here, the uniqueness refers not only to h but also to the interfaces. We observe that the self-similar solution **19** belongs to this class of solutions.

In Section 5, we shall construct a numerical scheme that yields solutions converging to the self-similar solution for large times, and it is natural to ask whether these solutions belong to the class introduced by Angenent. In particular this scheme yields solutions that are different from the ones obtained by the scheme in ref. 14. We conjecture that the solutions that we construct in the following section are physically relevant, but undoubtedly future research is needed to provide definite answers to the nonuniqueness question.

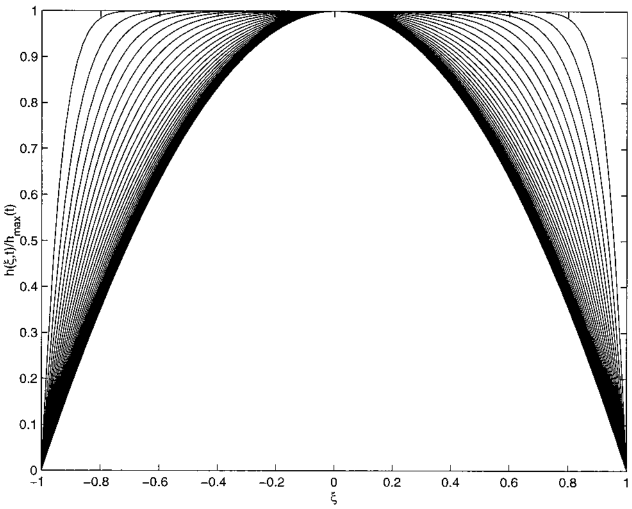

Fig. 1. The numerical solution to the Cauchy problem for c = 1.75 with the initial condition of a "smoothed block" type for different times in the scaled coordinates. The solution is collapsing to the self-similar asymptotics.

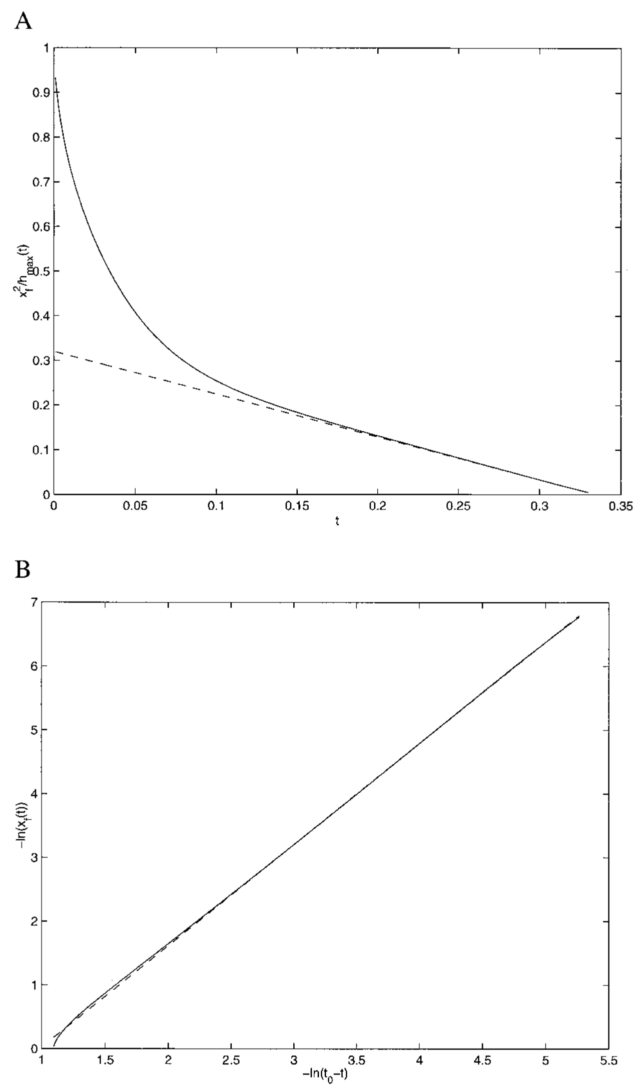

Fig. 2. (*A*) The determination of the parameter t_0. (*B*) The determination of the parameter *B*.

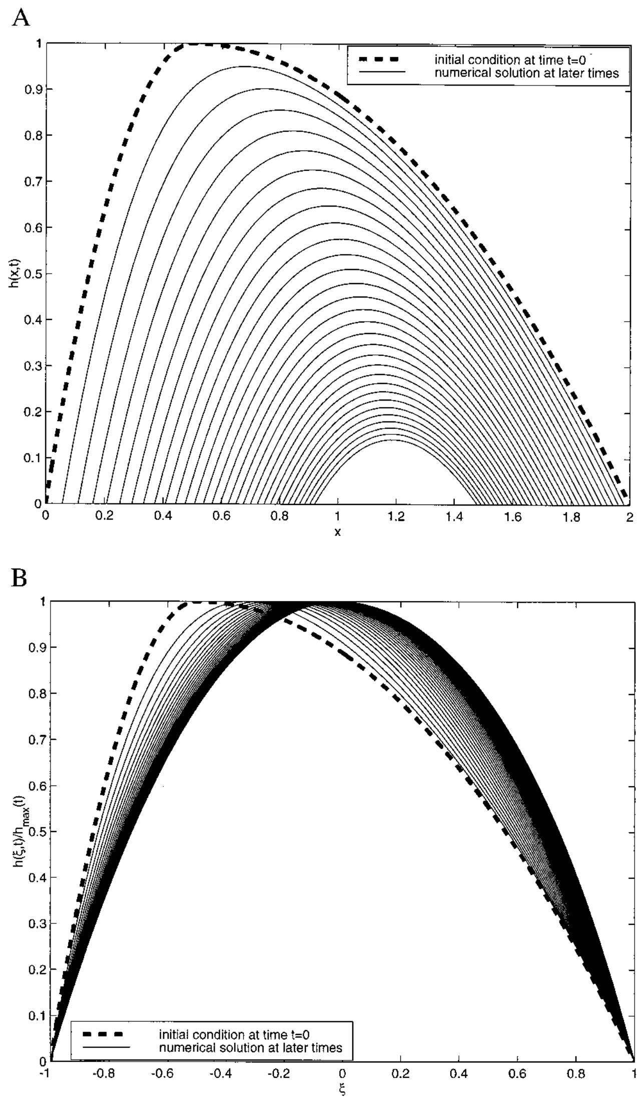

Fig. 3. (*A*) Evolution of the nonsymmetric initial distribution to a symmetric self-similar asymptotics. (*B*) The same evolution presented in scaled coordinates.

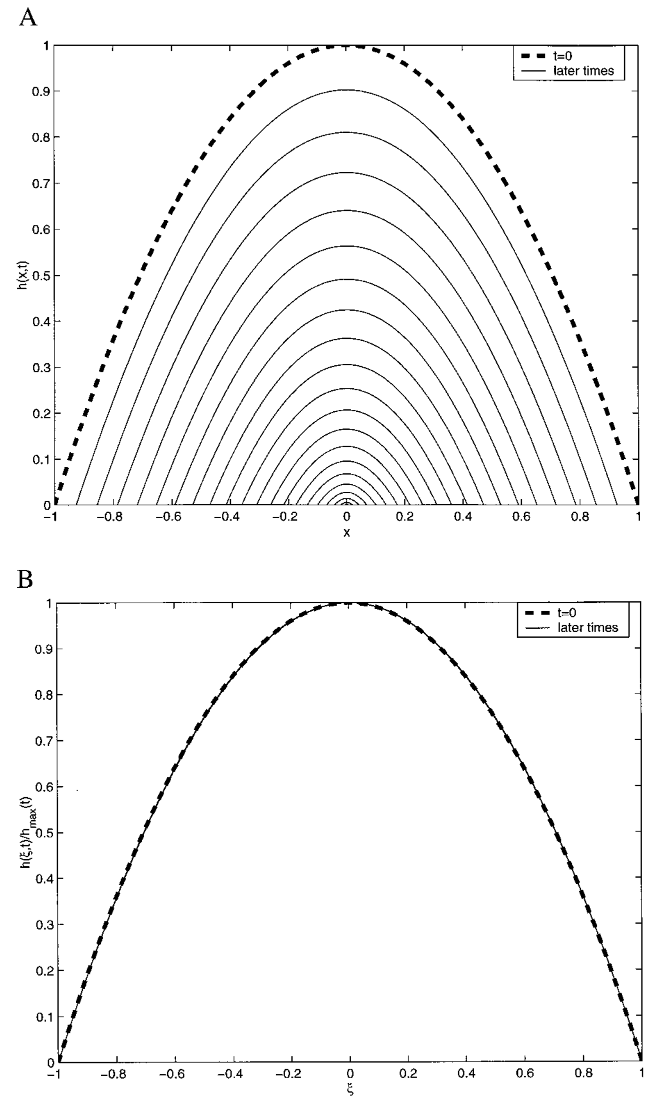

Fig. 4. The numerical solution preserves the self-similarity.

5. Numerical Experiment

The goal of the numerical experiment was to indicate that the self-similar solution obtained above attracts the solutions to non-self-similar Cauchy problems having the initial condition of compact support, generally speaking a non-symmetric one:

$$h(x, 0) = h_0(x), \qquad x_L(0) \le x \le x_R(0),$$

and $h(x, 0) \equiv 0$ outside the interval $x_L(0) \le x \le x_R(0)$. The basic equation **8** can be transformed to a form convenient for numerical calculations

$$\partial_t h = \frac{\dot{x}_0(t) - \dot{x}_f(t)\xi}{x_f(t)} \partial_\xi h + \frac{1}{[x_f(t)]^2}\left[h\partial^2_{\xi\xi}h - (c-1)(\partial_\xi h)^2\right],$$

where $\dot{x}_0 = dx_0/dt$, $\dot{x}_f = dx_f/dt$, and

$$\xi = \frac{x - x_0(t)}{x_f(t)}, \qquad x_f(t) = \frac{x_R(t) - x_L(t)}{2},$$

$$x_0(t) = \frac{x_R(t) + x_L(t)}{2},$$

so that the interval of new space variable ξ becomes fixed: $-1 \le \xi \le 1$, whereas the solution $h(x, t)$ is different from zero in the time-dependent interval $x_L(t) \le x \le x_R(t)$. Again, assuming naturally the quasi-steadiness of the level distribution in the vicinities of free boundaries, we obtain the conditions

$$\dot{x}_R(t) = (c-1)\frac{\partial_\xi h(1, t)}{x_f(t)}, \qquad \dot{x}_L(t) = (c-1)\frac{\partial_\xi h(-1, t)}{x_f(t)} \qquad \textbf{[26]}$$

at $t > 0$, and the basic equation takes the form:

$$\partial_t h = \frac{1}{x_f^2(t)}\left[(c-1)\partial_\xi h \frac{(\xi+1)\partial_\xi h(1, t) - (\xi-1)\partial_\xi h(-1, t)}{2} + h\partial_{\xi\xi}h - (c-1)(\partial_\xi h)^2\right] \qquad \textbf{[27]}$$

with the initial condition

$$h(\xi, 0) = h_0(\xi), \quad |\xi| \le 1; \qquad h(\xi, 0) \equiv 0, \quad |\xi| \ge 1. \qquad \textbf{[28]}$$

Two numerical schemes were used in our computations performed by finite-difference approximations: (*i*) a forward-in-time, centered-in-space *explicit* approximation, and (*ii*) a forward-in-time, centered-in-space *implicit* approximation. For the most part, numerical calculations have been run with the time step $\Delta t = 10^{-5}$ for the explicit scheme and $\Delta t = 10^{-4}$ for the implicit one. The number N of subintervals of length $\Delta\xi$, $N = 2/\Delta\xi$ was equal to 202 for both schemes. The results obtained by using these numerical approximations coincided with good accuracy. The absorption coefficient c was always equal to 1.75.

The first initial condition was taken as a "smoothed block": a homogeneous water level distribution smoothly going to zero at the edges. The results of the computation are presented in Fig. 1 in the form of the distribution of the scaled level: level $h(x, t)$ divided by maximum level at each time $h_{\max}(t)$. It is seen that the curves corresponding to different times collapse to the parabola, corresponding to the self-similar solution **18**. The time of collapse t_0 and the constant B were determined in the following way (Fig. 2): according to the intermediate asymptotics **19** at small $t_0 - t$,

$$\frac{x_f^2(t)}{h_{\max}(t)} = \frac{t_0 - t}{\mu F(0)}, \quad \textbf{[29]}$$

so that the quantity $x_f^2(t)/h_{\max}(t)$ should be a linear function of time, and the intersection of its graph with the time axis (Fig. 2A) gives the value of t_0. We obtain from **10** a linear relation between $\ln x_f$ and $\ln(t_0 - t)$, i.e., in the coordinates $-\ln x_f, -\ln(t_0 - t)$ a straight line with the slope μ. It gives us the value B (Fig. 2B) and an additional possibility of checking the asymptotics. Naturally t_0 and B depend on the initial condition. For our case, we found $t_0 = 0.345$, $B = 3.73$, and $\mu = 1.499$, which agrees well with the analytic value $\mu = 1.5$.

The next computation was performed for a nonsymmetric initial condition:

$$h(x, 0) = \begin{cases} -4x^2 + 4x, & 0 < x < \frac{1}{2} \\ -\frac{4}{9}x^2 + \frac{4}{9}x + \frac{8}{9}, & \frac{1}{2} < x < 2. \end{cases}$$

Figs. 3A and 3B demonstrate the behavior of the numerical solution for different times. It is clearly seen that the solution becomes symmetric and tends to the self-similar asymptotics (**19**). The values of $t_0 = 1.138$, $B = 0.63$, and $\mu = 1.5027$ have been calculated as before; the calculated and analytic values of μ agree with high precision.

For comparison we have taken the solution **19** for a certain t as an initial condition, and computed the solution to the partial differential equation further using the same algorithm. The results are presented in Fig. 4A for different times. Being plotted in scaled coordinates (Fig. 4B), they collapse to a single curve, giving us an additional check of the numerical procedure.

6. Conclusions

We presented a new derivation of the filtration-absorption equation based on a model of groundwater flow with partial absorption. It is shown that for a sufficiently large absorption constant the time of collapse is finite. A family of self-similar solutions to this equation is obtained. Numerical experiments indicate that these self-similar solutions obtained are self-similar intermediate asymptotics for the solutions to the Cauchy problems having the initial conditions data with compact support, but due to the nonuniqueness of the solution of the Cauchy problem future research is needed to provide more definite conclusions.

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