Convergence Analysis of the Particle Method for the Camassa-Holm Equation*

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Abstract

The purpose of this paper is to establish a new method for proving the convergence of the particle method applied to the Camassa-Holm (CH) equation. The CH equation is a strongly nonlinear, bi-Hamiltonian, completely integrable model in the context of shallow water waves. The equation admits solutions that are nonlinear superpositions of traveling waves that have a discontinuity in the first derivative at their peaks and therefore are called peakons. This behavior admits several diverse scientific applications, but introduce difficult numerical challenges. To accurately capture these solutions, one may apply the particle method to the CH equation. Using the concept of space-time bounded variation, we show that the particle solution converges to a global weak solution of the CH equation for positive Radon measure initial data.

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1 Introduction

The purpose of this paper is to establish convergence results for the particle method applied to the Camassa-Holm (CH) equation, given as

$$m_t + (um)_x + u_x m = 0 \qquad \text{with} \qquad m = u - \alpha^2 u_{xx}, \tag{1.1}$$

which is considered subject to the initial condition

$$m(x,0) = m_0(x). \tag{1.2}$$

Here the momentum m and velocity u are functions of the time variable t and the spatial variable x, and α is a length scale. Equation (1.1) arises in diverse scientific applications and, for instance, can be described as a bi-Hamiltonian model for waves in shallow water [3]. Equation (1.1) can also be used to quantify growth and other changes in shape, such as occurs in a beating heart, by providing the transformative mathematical path between the two shapes, (see, e.g, [8]).

The CH equation exhibits some remarkable properties. Of notable interest is the fact that the equation is completely integrable and yields peakon solutions which are solitons (whose identity is preserved through nonlinear interactions) with a sharp peak. Mathematically, this sharp peak is characterized by a discontinuity at the peak in the wave slope, and therefore are called peakons, [3].

Peakons may be accurately captured by applying particle methods to the CH equation as shown in, e.g., [4, 5, 6, 8]. In these methods, the solution is sought as a linear combination of Dirac distributions, whose positions and coefficients represent locations and weights of the particles, respectively. The solution is then found by following the time evolution of the locations and the weights of these particles according to a system of ODEs obtained by considering a weak formulation of the problem. The main advantage of particle methods is their (extremely) low numerical diffusion that allows one to capture a variety of nonlinear waves with high resolution, see, e.g., [9] and references therein.

In this paper, we apply the particle method for numerical solution of the CH equation. We begin with a brief overview of the particle method and some of its main features which are relevant to our discussion. The main analytical results we provide is the convergence proof of the particle method. While previous convergence results have been established for this equation (e.g. see [1, 2, 4, 7, 10]), we propose a new self-contained method for showing the convergence by establishing BV estimates for the particle solution. We then verify that both the particle solution and its limit are weak solutions to the CH equation to complete our study on the convergence analysis.

2 Description of a Particle Method

To solve (1.1) by the particle method, we follow the method introduced in [6]. That is, we look for a weak solution in the form of a linear combination of Dirac delta distributions,

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t))). \tag{2.1}$$

Here, $x_i(t)$ and $p_i(t)$ represent the location of the i-th particle and its weight, and N denotes the total number of particles. The solution is then found by following the time evolution of the locations and the weights of the particles according to the following system of ODEs [6]:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = u^N(x_i(t), t), \\ \dfrac{dp_i(t)}{dt} + u_x^N(x_i(t), t)p_i(t) = 0. \end{cases} \tag{2.2}$$

Using the special relationship between m and u given in (1.1), one can directly compute the velocity u and its derivative, by the convolution $u^N = G * m^N$, where G is the Green's function

$$G(|x-y|) = \frac{1}{2\alpha} e^{-|x-y|/\alpha}, \tag{2.3}$$

associated with the one dimensional Helmholtz operator in (1.1). Thus we have the following exact expressions for both $u(x,t)$ and (by direct computation) $u_x(x,t)$:

$$u^N(x,t) = \frac{1}{2\alpha} \sum_{i=1}^{N} p_i(t) e^{-|x-x_i(t)|/\alpha}, \tag{2.4}$$

$$u_x^N(x,t) = -\frac{1}{2\alpha^2} \sum_{i=1}^{N} p_i(t) \text{sgn}(x - x_i(t)) e^{-|x-x_i(t)|/\alpha}. \tag{2.5}$$

In practice, except for very special cases, the functions $x_i(t)$ and $p_i(t),\ i = 1, \dots, N$ have to be determined numerically and the system (2.2) must be integrated by an appropriate ODE solver. In order to initiate the time integration, one should choose the initial positions of particles, x_i^0, and the weights, p_j^0, so that (2.1) represents a high-order approximation to the initial data $m_0(x)$ in (1.2), as it is shown in [6, 9].

The system (2.2) may be derived in one of two ways. Following [6], we may consider a weak formulation of the problem and make a suitable substitution to derive (2.2) or one may follow [4, 5] by considering the Hamiltonian structure of (1.1). The latter property of the particle system and its complete integrability allows one to establish the global existence results for the solution of (2.2) and to show that for a relatively wide class of initial data there are no particle collisions in finite times [5, Proposition 2.3].

3 Convergence Results

In this section,we show that the particle method given by (2.1) and (2.2) converges to a weak solution of the CH equation (1.1). While others have shown this result in a variety of ways, we propose a self-contained concise way of showing convergence by establishing BV estimates for u^N and u_x^N. Once these estimates are established, we use the compactness result, associated with BV functions, to pass to the limit and show that the particle solution associated with the CH equation converges to a weak solution of (1.1). Throughout this section, we shall assume that the initial momenta $p_i(0)$ are positive and that there are no particle collisions in finite time. It should also be noted that the total momentum of the particle system is conserved as it was shown in [6].

3.1 Space and Time BV Estimates

In what follows, we recall the definition of the total variation of a function and prove that the total variations of both $u^N(x,t)$ and $u_x^N(x,t)$ are bounded.

Definition 3.1. Consider a (possibly unbounded) interval $J \subseteq \mathbb{R}$ and a function $u : J \to \mathbb{R}$. The total variation of u is defined as

$$\text{Tot.Var.}\,\{u\} \equiv \sup\left\{\sum_{j=1}^{N} |u(x_j) - u(x_{j-1})|\right\}, \tag{3.1}$$

where the supremum is taken over all $N \geq 1$ and all $(N+1)$-tuples of points $x_j \in J$ such that $x_0 < x_1 < \cdots < x_N$. If the right hand side of (3.1) is bounded, then we say that u has bounded variation, and write $u \in BV(\mathbb{R})$.

The following theorem establishes the necessary space and time BV estimates.

Theorem 3.2. *Let $u^N(x,t)$ and $u_x^N(x,t)$ be functions defined in* (2.4) *and* (2.5)*, respectively. Then, both $u^N(x,t)$ and $u_x^N(x,t)$ are BV functions in the two variables x,t.*

Proof. We begin with showing that Tot. Var. $\{u^N(\cdot,t)\}$ and Tot. Var. $\{u_x^N(\cdot,t)\}$ are bounded. Indeed, from (2.3) we have Tot.Var.$\{G(x)\} = 1/\alpha$ and Tot.Var.$\{G_x(x)\} = 2/\alpha^2$. Using the fact that the total momentum of the particle system is conserved, we obtain

$$\text{Tot.Var.}\{u^N(\cdot,t)\} \le \sum_{j=1}^{N} p_j(t)\text{Tot.Var.}\{G(x)\} = \frac{1}{\alpha}\sum_{j=1}^{N} p_j(t) = \frac{1}{\alpha}|m_0|, \tag{3.2}$$

$$\text{Tot.Var.}\{u_x^N(\cdot,t)\} \le \sum_{j=1}^{N} p_j(t)\text{Tot.Var.}\{G_x(x)\} = \frac{2}{\alpha^2}\sum_{j=1}^{N} p_j(t) = \frac{2}{\alpha^2}|m_0|. \tag{3.3}$$

In order to prove that $u^N(x,t)$ and $u_x^N(x,t)$ have bounded variation with respect to t as well, it now suffices to show that u^N and u_x^N are both Lipschitz continuous in time in L^1, [1, Theorem 2.6].

We first consider expression (2.4) for u^N to have

$$\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\,dx \le \frac{1}{2\alpha}\int_{-\infty}^{\infty}\sum_{i=1}^{N} p_i(t)\left|e^{-|x-x_i(t)|/\alpha} - e^{-|x-x_i(s)|/\alpha}\right| dx$$
$$+\frac{1}{2\alpha}\int_{-\infty}^{\infty}\sum_{i=1}^{N} e^{-|x-x_i(s)|/\alpha}|p_i(t)-p_i(s)|\,dx.$$

Simple calculations show that

$$\int_{-\infty}^{\infty}\left|e^{-|x-x_i(t)|/\alpha} - e^{-|x-x_i(s)|/\alpha}\right| dx \le 4|x_i(t)-x_i(s)| \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-|x-x_i(t)|/\alpha}\,dx \le 2\alpha,$$

and hence, we have

$$\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\,dx \le \frac{2}{\alpha}\sum_{i=1}^{N} p_i(t)|x_i(t)-x_i(s)| + \sum_{i=1}^{N}|p_i(t)-p_i(s)|. \tag{3.4}$$

Using the ODE system (2.2), it now follows from (3.4) that

$$\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\,dx \le \frac{3}{2\alpha^2}|m_0|^2\,(t-s)\,,$$

proving that u^N is Lipshitz continuous in time in L^1 and thus has bounded variation with respect to both x and t.

Similarly, for u_x^N we have:

$$\int_{-\infty}^{\infty} |u_x^N(x,t) - u_x^N(x,s)|\,dx \le \frac{3}{2\alpha^3}|m_0|^2(t-s),$$

which together with (3.3) proves that $u_x^N(x,t)$ is a BV function in x, t and the statement of the theorem. □

3.2 Compactness and Convergence

In this section, we use compactness properties associated with BV functions to prove convergence of the particle method. To this end, we first give a definition of a weak solution of the CH equation (1.1) and show that the particle solution (m^N, u^N) obtained from (2.1), (2.2) is indeed a weak solution of the CH equation.

Definition 3.3. $u(x,t) \in C(0,T;H^1(\mathbb{R})), m(x,t) = u(x,t) - \alpha^2 u_{xx}(x,t)$ is said to be a weak solution to (1.1) if

$$\int_{-\infty}^{\infty} \phi(x,0)m(x,0)\,dx + \int_0^{\infty}\int_{-\infty}^{\infty} \left[\phi_t(x,t) - \alpha^2\phi_{txx}(x,t)\right] u(x,t)\,dxdt + \int_0^{\infty}\int_{-\infty}^{\infty} \left[\frac{3}{2}\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)\right] u^2(x,t)\,dxdt + \int_0^{\infty}\int_{-\infty}^{\infty} \frac{\alpha^2}{2}\phi_x(x,t)u_x^2(x,t)\,dxdt = 0 \tag{3.5}$$

for all $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$.

Proposition 3.4. *Assume that $m_0 \in \mathcal{M}(\mathbb{R})$, then the particle solution $(m^N(x,t), u^N(x,t))$ given by* (2.1), (2.2) *is a weak solution of the problem* (1.1), (1.2).

Proof. Let $m^N(x,0), m^N(x,t)$ and $u^N(x,t), u_x^N(x,t)$ be given by formulae (2.1) and (2.2), respectively, and $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ be a test function. Then, the following relations can be easily established by direct substitutions:

$$\langle m^N, \phi_t\rangle = \langle u^N, \phi_t - \alpha^2\phi_{txx}\rangle, \tag{3.6}$$

$$\langle m^N u^N, \phi_x\rangle = \left\langle (u^N)^2, \phi_x - \frac{\alpha^2}{2}\phi_{xxx}\right\rangle + \alpha^2\left\langle (u_x^N)^2, \phi_x\right\rangle, \tag{3.7}$$

$$\left\langle m^N u_x^N, \phi\right\rangle = \left\langle \frac{\alpha^2(u_x^N)^2 - (u^N)^2}{2}, \phi_x\right\rangle. \tag{3.8}$$

Using (3.6)–(3.8) and substituting $m^N(x,t)$ as defined by (2.1) into (3.5), yields

$$\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) + \int_0^\infty \sum_{i=1}^{N} p_i(t)\phi_t\left(x_i(t),t\right)\,dt +$$
$$\int_0^\infty \sum_{i=1}^{N} p_i(t)u^N(x_i(t),t)\phi_x\left(x_i(t),t\right)\,dt - \int_0^\infty \sum_{i=1}^{N} p_i(t)u_x^N\left(x_i(t),t\right)\phi\left(x_i(t),t\right)\,dt = 0. \tag{3.9}$$

We now add and subtract $\sum_{i=1}^{N}\int_0^\infty p_i(t)\,\dfrac{dx_i}{dt}\phi_x(x_i(t),t)\,dt$ into the last equation, use the fact that

$$\frac{d\phi(x_i(t),t)}{dt} = \phi_x(x_i(t),t)\frac{dx_i(t)}{dt} + \phi_t(x_i(t),t)$$

and rewrite (3.9) as follows:

$$\begin{aligned}
&\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) + \int_0^\infty \sum_{i=1}^{N} p_i(t)\frac{d\phi(x_i(t),t)}{dt}\,dt \\
&\int_0^\infty \sum_{i=1}^{N} p_i(t)\left[u^N(x_i(t),t) - \frac{dx_i(t)}{dt}\right]\phi_x\left(x_i(t),t\right)\,dt \\
&-\int_0^\infty \sum_{i=1}^{N} p_i(t)u_x^N\left(x_i(t),t\right)\phi\left(x_i(t),t\right)\,dt = 0.
\end{aligned} \tag{3.10}$$

Integrating by parts the second term in the first row in (3.10), and rearranging other terms, we finally obtain:

$$\begin{aligned}
&\int_0^\infty \sum_{i=1}^{N} p_i(t)\left[\frac{dx_i(t)}{dt} - u^N(x_i(t),t)\right]\phi_x\left(x_i(t),t\right)\,dt \\
&+\int_0^\infty \sum_{i=1}^{N} \left[\frac{dp_i(t)}{dt} + p_i(t)u_x^N\left(x_i(t),t\right)\right]\phi(x_i(t),t)\,dt = 0.
\end{aligned} \tag{3.11}$$

Since the functions $x_i(t)$ and $p_i(t)$ satisfy the system (2.2), the last equation holds for any ϕ implying that m^N, u^N defined by (2.1), (2.4) is a weak solution of (1.1), (1.2). This completes the proof. □

We are now in a position to establish a convergence result for the particle method. Using the BV estimates for $u^N(x,t)$ and $u_x^N(x,t)$, and the fact that the particle solution is a weak solution to the CH equation, we may establish the following convergence result.

Theorem 3.5. *Suppose that $(m^N(x,t), u^N(x,t))$ is a particle solution of* (2.1), (2.2) *with initial approximation $m^N(\cdot,0) \overset{*}{\rightharpoonup} m_0$, $m_0 \in \mathcal{M}_+(\mathbb{R})$. Then there exist functions $u(x,t) \in BV(\mathbb{R}\times\mathbb{R}_+)$ and $m(x,t) \in \mathcal{M}_+(\mathbb{R}\times\mathbb{R}_+)$ such that $m^N(x,t)$ and $u^N(x,t)$ converge to $m(x,t)$ and $u(x,t)$, respectively in the sense of distributions as $N \to \infty$. Furthermore, the limit (u,m) is the unique weak solution to the CH equation* (1.1) *with regularity $u \in C(0,T;H^1(\mathbb{R}))$, $u_x \in BV(\mathbb{R}\times\mathbb{R}_+)$.*

Proof. Using BV estimates for $u^N(x,t)$ and $u^N_x(x,t)$, we refer to the compactness property in [1, Theorem 2.4] and conclude that there exist functions u and u_x and a subsequence (still labeled as $u^N(x,t)$) such that

$$||u^N - u||_{L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)} \to 0, \qquad ||u^N_x - u_x||_{L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)} \to 0 \tag{3.12}$$

as $N \to \infty$.

From Proposition 3.4, we know that (m^N, u^N) is a weak solution and thus satisfy equation (3.5). To complete the proof, we need to show that each terms in (3.5) converges to that of the limit solution (m,u). Indeed, by the construction of the initial approximation, one has

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\phi(x,0)m^N(x,0)\,dx = \int_{-\infty}^{\infty}\phi(x,0)m(x,0)\,dx \tag{3.13}$$

Furthermore, for any $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, we have

$$\begin{aligned}&\Big|\iint \phi\left((u^N)^2 - (u)^2\right)dxdt\Big| = \Big|\iint \phi(u^N+u)(u^N-u)dxdt\Big| \\ &\le \|\phi\|_{L^\infty}(\|u^N\|_{L^\infty} + \|u\|_{L^\infty})\iint_{(x,t)\in\text{supt of }\phi}\left|u^N(x,t)-u(x,t)\right|dxdt \to 0,\end{aligned}$$

and thus

$$\iint \phi(u^N)^2 dxdt \to \iint \phi u^2 dxdt,$$

as as $N \to \infty$. Similarly, we obtain

$$\iint \phi(u^N_x)^2 dxdt \to \iint \phi(u_x)^2 dxdt.$$

This shows that the limit (m,u) is indeed a weak solution to the CH equation. □

4 Conclusion

In this paper, we have provided a new way of showing that for positive Radon measure initial data, the particle method applied to the CH equation will converge to the unique global weak solution. To this extent, we have only highlighted some of the main results obtained from our study. A full version of this paper will be published in the future and will provide additional details and some verification of results that were omitted above. Furthermore, numerical experiments will be performed that illustrates the no-crossing behavior of the solutions to the CH equation as well as the soliton behavior obtained by the complete integrability of the CH equation.

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