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CONVERGENCE OF A PARTICLE METHOD AND GLOBAL WEAK SOLUTIONS OF A FAMILY OF EVOLUTIONARY PDES*

ALINA CHERTOCK[†], JIAN-GUO LIU[‡], AND TERRANCE PENDLETON[†]

Abstract. The purpose of this paper is to provide global existence and uniqueness results for a family of fluid transport equations by establishing convergence results for the particle method applied to these equations. The considered family of PDEs is a collection of strongly nonlinear equations which yield traveling wave solutions and can be used to model a variety of flows in fluid dynamics. We apply a particle method to the studied evolutionary equations and provide a new self-contained method for proving its convergence. The latter is accomplished by using the concept of space-time bounded variation and the associated compactness properties. From this result, we prove the existence of a unique global weak solution in some special cases and obtain stronger regularity properties of the solution than previously established.

Key words. Camassa–Holm equation, Degasperis–Procesi equation, Euler–Poincaré equation, global weak solution, particle method, space-time BV estimates, peakon solutions, conservation laws, completely integrable systems

AMS subject classifications. 35B65, 35C08, 35D05, 35L65, 37K10, 65M25, 74J35, 76B15

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1. Introduction.

The purpose of this paper is to apply a particle method to a family of evolutionary PDEs and use the convergence properties of the method for establishing global existence and uniqueness results for the considered equations. In the one-dimensional (1-D) case, the equations read as follows:

$$m_t + (um)_x + (b-1)mu_x = 0, \qquad u = G * m, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

with $b > 1$ and subject to the initial condition

$$m(x,0) = m_0(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Here, the momentum m and velocity u are functions of a time variable t and spatial variable x, and $G(x)$ is the Green's kernel.

Equation (1.1) admits traveling wave solutions of the form $u(x,t) = aG(x-ct)$, with speed $c = -aG(0)$, proportional to the solution amplitude. The bifurcation parameter b in (1.1) gives the relation of the stretching, bu_xm, to convection, um_x, and provides a balance for the nonlinear solution behavior. The kernel $G(x)$ relates the velocity with the momentum density through the convolution product

$$u(x,t) = u = G * m = \int_{\mathbb{R}} G(x-y)m(y,t)\,dy \tag{1.3}$$

and determines the shape of the traveling wave and the length scale for (1.1); see, e.g., [44].

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[†]Department of Mathematics, North Carolina State University, Raleigh, NC 27695 (chertock@math.ncsu.edu, tlpendle@ncsu.edu). The work of these authors was supported in part by NSF grant DMS-0712898.

[‡]Department of Physics and Mathematics, Duke University, Durham, NC 27708 (Jian-Guo.Liu@duke.edu). The work of this author was supported in part by NSF grant DMS 10-11738.

The family of evolutionary PDEs given by (1.1), (1.2) arises in diverse scientific applications and enjoys several remarkable properties both in the 1-D and multidimensional cases. For example, in one dimension, for the specific choice of

$$G(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}, \tag{1.4}$$

(1.1) becomes

$$m_t + (um)_x + (b-1)mu_x = 0, \qquad m = u - \alpha^2 u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.5}$$

where α is a positive constant and $G(x)$ is the Green function associated with the modified Helmholtz operator in (1.5). Equation (1.5) coincides with the dispersionless case of the Camassa–Holm (CH) equation for shallow water if $b = 2$ (see [5, 22, 23, 27, 35]) and with the Degasperis–Procesi (DP) equation used to model the propagation of nonlinear dispersive waves if $b = 3$; see [30]. In this special case of (1.5), the corresponding traveling wave solutions assume the form $u(x,t) = ae^{-|x-ct|}$ with speed c, amplitude a, and length α. The solutions are characterized by a discontinuity in the first derivative at their peaks and are thus referred to as peakon solutions. Both CH and DP equations are completely integrable as Hamiltonian systems and their peakon solutions are true solitary waves that emerge from the initial data. Peakons for either $b = 2$ or $b = 3$ exhibit a remarkable stability—their identity is preserved through nonlinear interactions; see, e.g., [5, 6] and [28, 29, 30, 48, 52]. Peakons are also orbitally stable, i.e., their shape is maintained under small perturbations; see, e.g., [26, 32, 46]. We note that peakons can also be considered as waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves; see [18, 21, 58]. For a more complete discussion on the hydrodynamical properties of peakons generated from the CH or DP equation, see [24, 45].

The two-dimensional (2-D) version of (1.5) with $b = 2$, the so-called EPDiff equation (Euler–Poincaré equation associated with the diffeomorphism group) appears in the theory of fully nonlinear shallow water waves [9, 41, 42, 43, 44]. Applying viscosity to the incompressible, three-dimensional analogue of this equation produces the Navier–Stokes α-model for the averaged fluid equations (see, e.g., [10]). The equation (1.1) has many further interpretations beyond fluid applications. For instance, in 2-D, it coincides with the averaged template matching equation (ATME) for computer vision (see, e.g., [36, 39, 40]). One could also use (1.1) to quantify growth and other changes in shape, such as occurs in a beating heart, by providing the transformative mathematical path between the two shapes (see, e.g., [41]).

The Cauchy problems for both the CH ($b = 2$) and DP ($b = 3$) equations have been extensively studied in the literature. We refer the reader to a review paper [54], where a survey of recent results on well-posedness and existence of local and global weak solutions for the CH equation is presented. In particular, the local well-posedeness results for the CH equation in $H^s(\mathbb{R}), s > 3/2$, were established in [19, 47, 57]. The continuation of solutions to the CH equation after wave breaking in $L^\infty\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$ was established in [4, 3]. The existence of a global weak solution to the CH equation in $L^\infty\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$ was proved in [3, 20, 59], and in [25] it was shown that this global solution is unique.

Recent results related to well-posedness and existence of local and global weak solutions of the DP equation can be found, e.g., in [16, 33, 51, 61, 62], where it was proved that the global weak solutions of the DP equation belong to$L^\infty\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$ and global entropy weak solutions are in $L^\infty(\mathbb{R}_+, L^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R}))$ and $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}) \cap$

$L^4(\mathbb{R})$). The local well-posedeness and several global existence results were obtained in [34] for a general case of the initial-value problem (IVP) (1.5), (1.2) with different values of the parameter b.

Capturing peakon solutions numerically poses quite a challenge—especially if one considers a peakon-antipeakon interaction. Several numerical methods have been proposed for simulating peakon interactions for the CH equation such as finite-difference, finite-element, and spectral methods [1, 17, 37, 38, 56, 60]. A few numerical methods, such as conservative finite-difference schemes, have been used to study the DP equation (see [53]). Many of these methods are computationally intensive and require very fine grids along with adaptivity techniques in order to simulate the peakon behavior.

Solutions of (1.1), (1.2) can be accurately captured by using a particle method, as shown in [7, 8, 11, 15] for the CH equation and in [11] for the EPDiff equation. In the particle method, described in [11, 15], the solution is sought as a linear combination of Dirac distributions, whose positions and coefficients represent locations and weights of the particles, respectively. The solution is then found by following the time evolution of the locations and the weights of these particles according to a system of ODEs obtained by considering a weak formulation of the problem. The particle methods presented in [7, 8] have been derived using a discretization of a variational principle and provide the equivalent representation of the ODE particle system. The main advantage of particle methods is their (extremely) low numerical diffusion that allows one to capture a variety of nonlinear waves with high resolution; see, e.g., [12, 13, 14, 55] and references therein.

A convergence analysis for the particle method applied to the CH equation was studied in [5] and [15]. In [5], the authors used the Hamiltonian structure of the CH equation and its complete integrability to establish error estimates for the particle method when the solutions are smooth. In [15], the convergence of the particle method for the CH equation has been proved using the concept of space-time bounded variation. Properties of the particle method were also studied in the context of the DP equation in [28, 29, 30, 44].

In this paper, we apply the particle method from [11, 15] to IVP (1.1), (1.2) and propose a new self-contained proof of its convergence for any $b > 1$ by establishing bounded variation (BV) estimates of the particle solution and using the associated compactness properties [49, 50]. To this end, we assume that the kernel $G(x)$ in (1.1) satisfies the the following properties:

- (I) $G(x)$ is an even function, that is, $G(-x) = G(x)$ for any $x \in \mathbb{R}$.
- (II) $G(x) \in C^1(\mathbb{R}\backslash 0)$, $||G||_\infty = G(0)$, and $G'(0) = 0$.
- (III) $G(x), G'(x) \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and consequently both $||G||_\infty$ and $||G'||_\infty$ are bounded.

From this convergence result, we provide a novel method for obtaining global existence and uniqueness results for (1.5), (1.2) with $b > 1$ and $G(x)$ given by (1.4) and show that the global weak solution of (1.5), (1.2) has stronger regularity properties than those previously established in, e.g., [34].

The paper is organized as follows. We begin in section 2 with a brief overview of the particle method and some of its main features relevant to our discussion. We then show that both the particle solution and its derivative are functions of BV for any $b > 1$ and an arbitrary kernel G satisfying properties (I) through (III). Section 3 is dedicated to the special case of IVP (1.5), (1.2) with $b > 1$ and G given by (1.4). In particular, in section 3.1, we prove that for a relatively wide class of initial data there exists a unique global solution of the particle ODE system. Next, in section 3.2, we use the compactness results associated with BV functions and verify that both the particle solution and its limit are weak solutions to the b-equation, and we complete our study on the convergence analysis. Finally, in section 3.3, we use our convergence results and the obtained BV estimates to prove the existence of a unique global weak solution for the b-family of fluid transport equations (1.5), (1.2) for any $b > 1$. The appendix provides additional details and proofs that were omitted in the main text.

2. Particle method. In this section, we describe the particle method and show how it is used to solve the b-family of fluid transport equations. We also establish important conservation properties of the corresponding particle system and obtain BV estimates of the particle solution that will allow us to prove (in section 3) our main result—existence of a global weak solution for the IVP (1.5), (1.2).

2.1. Description of the particle method. To solve the b-equation by a particle method, we follow [11, 15] and search for a weak solution of (1.1) as a linear combination of Dirac delta functions:

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t)). \tag{2.1}$$

Here, $x_i(t)$ and $p_i(t)$ represent the location of the ith particle and its weight, and N denotes the total number of particles. The locations and weights of the particles are then evolved in time according to the following system of ODEs, obtained by substituting (2.1) into a weak formulation of (1.1) (for a detailed derivation of the ODE system see [11]):

$$\begin{cases} \dfrac{dx_i(t)}{dt} = u^N(x_i(t),t), \\ \dfrac{dp_i(t)}{dt} + (b-1)u_x^N(x_i(t),t)p_i(t) = 0. \end{cases} \tag{2.2}$$

Using the special relationship between m and u given in (1.1), one can explicitly compute the velocity u and its derivative by the convolution $u^N = G * m^N$. Thus we have the following exact expressions for both $u^N(x,t)$ and $u_x^N(x,t)$:

$$u^N(x,t) = \sum_{i=1}^{N} p_i(t)G(x - x_i(t)), \tag{2.3}$$

$$u_x^N(x,t) = \sum_{i=1}^{N} p_i(t)G'(x - x_i(t)). \tag{2.4}$$

With the exception of a few isolated cases, the functions $x_i(t)$ and $p_i(t)$, $i = 1,\dots,N$, must be determined numerically and the system (2.2) must be integrated by choosing an appropriate ODE solver. In order to start the time integration, one should choose the initial positions of particles, x_i^0, and the weights, p_i^0, so that (2.1) represents a high-order approximation to the initial data $m_0(x)$ in (1.2), as shown in [11, 55]. The latter can be done in the sense of measures on $\mathbb{R}$. Namely, we choose $(x_i(0), p_i(0))$ in such a way such that for any test function $\phi(x) \in C_0^\infty(\mathbb{R})$, we have that

$$\int_{\mathbb{R}} m_0(x)\phi(x)\,dx \approx \langle m^N(\cdot,0), \phi(\cdot)\rangle = \sum_{i=1}^{N} p_i(0)\phi(x_i), \tag{2.5}$$

where

$$m^N(x,0) = m_0^N(x) = \sum_{i=1}^{N} p_i(0)\delta(x - x_i(0)). \tag{2.6}$$

Based on (2.5), we observe that determining the initial weights, p_i^0, is exactly equivalent to solving a standard numerical quadrature problem. One way of solving this problem is to first divide the computational domain Ω into N nonoverlapping subdomains Ω_i such that their union is Ω. We then set the ith particle $x_i(0)$ to be the center of mass Ω_i. For instance, given initial particles $\{x_i(0)\}_{i=1}^N$, we may define Ω_i as

$$\Omega_i = [x_{i-1/2}, x_{i+1/2}] = \left\{x \mid x_{i-1/2} \le x \le x_{i+1/2}\right\}, \quad i = 1, \dots, N,$$

and by $x_i(0) = \Delta x$ the center Ω_i. For example, a midpoint quadrature will then be given by setting $p_i(0) = \Delta x\, m_0(x_i(0))$.

In general, one can build a sequence of basis functions $\{\sigma_i(x)\}_{i=1}^N$ that will aid in solving the numerical quadrature problem given by (2.5). Indeed, we have the following proposition.

PROPOSITION 2.1. *Let $\chi(x)$ be a characteristic function,*

$$\chi_{\Omega_i}(x) = \begin{cases} 1 & \text{when } x \in \Omega_i, \\ 0 & \text{when } x \in X \setminus \Omega_i, \end{cases} \qquad \sum_{i=1}^{N} \chi_{\Omega_i} = 1,$$

and let $\sigma(x) \in C_0^\infty(\mathbb{R})$ be a mollifier, that is,

$$\sigma(x) \ge 0, \quad \int_{\mathbb{R}} \sigma(x)\, dx = 1, \quad \lim_{\epsilon\to 0} \sigma_\epsilon(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\sigma(x/\epsilon) = \delta(x).$$

Then

$$1 = 1 * \sigma_\epsilon = \sum_{i=1}^{N} \chi_{\Omega_i} * \sigma_\epsilon = \sum_{i=1}^{N} \sigma_i(x).$$

From here one can approximate the initial data by taking $p_i(0) = \int_{\mathbb{R}} \sigma_i(x)\, dm_0$ in (2.6). We note that the latter makes sense only if $m_0 \in \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the set of Radon measures. Furthermore, one can prove that m_0^N converges weakly to $m_0(x)$ as $N \to \infty$. Indeed, given the above definition for $p_i(0)$, one can show that if m_0^N is given by (1.2), then m_0^N converges weakly to m_0 in the sense of measures.

PROPOSITION 2.2. *Let $m_0(x)$ be defined by (1.2) and $m_0^N(x)$ be given by (2.6). Let $h = \max_{1\le i\le N} |x_{i+1} - x_i|$. Then m_0^N converges weakly to $m_0(x)$ in the sense of measures.*

Proof. For any $\phi(x) \in C_0^\infty(\mathbb{R})$, we denote $M_0 = \int_{\mathbb{R}} dm_0$ and have the following:

$$\begin{aligned}
\left| \int_{\mathbb{R}} \phi(x)\, dm_0 - \int_{\mathbb{R}} \phi(x)\, dm_0^N \right| &= \left| \sum_{i=1}^{N} \left(\int_{\mathbb{R}} \phi(x)\sigma_i(x)\, dm_0 - \phi(x_i) \int_{\mathbb{R}} \sigma_i(x)\, dm_0 \right) \right| \\
&= \left| \sum_{i=1}^{N} \int_{\mathbb{R}} \left(\phi(x) - \phi(x_i)\right) \sigma_i(x)\, dm_0 \right| \\
&\le Kh \sum_{i=1}^{N} \int_{\mathbb{R}} \sigma_i(x)\, dm_0 = KhM_0 \to 0
\end{aligned}$$

as $h \to 0$ or equivalently as $N \to \infty$. $\square$

2.2. Properties of the particle system. We now discuss some general properties of the derived particle method. In particular, we establish conservation laws for the particle momenta and show that the particles propagate with a finite speed.

First, we prove the conservation property of the particle system.

PROPOSITION 2.3. *The total momentum of the particle system* (2.2) *is conserved. That is,*

$$\frac{d}{dt}\left[\sum_{i=1}^{N} p_i(t)\right] = 0. \tag{2.7}$$

Proof. We recall (2.2) and (2.4) to obtain

$$\frac{d}{dt}\left[\sum_{i=1}^{N} p_i(t)\right] = -\sum_{i=1}^{N}\sum_{j=1}^{N}(b-1)p_i(t)p_j(t)G'(x_i(t)-x_j(t)). \tag{2.8}$$

Taking into account the fact that $G'(x)$ is an odd function and $G'(0) = 0$ (see properties (I) and (II) of G) and the fact that the summation in (2.8) is performed over all $i, j = 1, \ldots, N$, we obtain (2.7) and consequently

$$\sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} p_i(0) = M_0. \qquad \square \tag{2.9}$$

Next, we assume that $x_1(0) < \cdots < x_N(0)$ and $p_i(0) > 0,\ i = 1, \ldots, N$, and show that these properties are preserved by the flow. We also provide an estimation for the speed of propagation of particles.

PROPOSITION 2.4. *Suppose that the initial momenta in* (2.6) *are positive, i.e.,* $p_i(0) > 0$ *for all* $i = 1, \ldots, N$*. Then* $p_i(t) > 0$ *for all* $i = 1, \ldots, N$ *and* $t > 0$*.*

Proof. The proof follows directly from [8], in which one may use the fact that the total momentum is conserved; see (2.9) as well as Gronwall's inequality to obtain

$$p_i(0)e^{-Kt} \le p_i(t) \le p_i(0)e^{Kt}, \qquad i = 1, \ldots, N, \tag{2.10}$$

where $K = (b-1)M_0||G'||_\infty$. We observe that the left inequality prevents $p_i(t)$ from being negative as t goes to 0, while the right inequality prevents $p_i(t)$ from being negative as t goes off to infinity. Hence, $p_i(t) > 0$ for all $i = 1, \ldots, N$ and $t > 0$. $\square$

PROPOSITION 2.5. *Suppose that* $\frac{dx_i(t)}{dt}$ *is given by* (2.2) *in the interval* $0 \le t \le T$*. Then there exists a constant* $0 < C \le \infty$ *such that*

$$|x_i(t)| < CT. \tag{2.11}$$

Proof. From (2.2), we have the following:

$$\left|\frac{dx_i(t)}{dt}\right| = |u^N(x_i(t), t)| = \left|\sum_{j=1}^{N} p_j(t)G(|x_j(t) - x_i(t)|)\right| \le C. \tag{2.12}$$

The last inequality holds due to the conservation of total momentum (2.9) and properties (I) through (III) of $G(x)$. Integrating both sides of (2.12) over $0 \le t \le T$ leads us to the desired conclusion (2.11). $\square$

Remark 2.6. It should be observed that the time-dependent parameters $x_i(t)$ and $p_i(t)$ in (2.2) satisfy the following dynamics equations [44]:

$$\frac{dx_i}{dt} = \frac{\partial H^N}{\partial p_i}, \qquad \frac{dp_i}{dt} = -(b-1)\frac{\partial H^N}{\partial x_i}, \qquad i = 1, \dots, N, \tag{2.13}$$

where the function $H^N(t)$ is defined as

$$H^N(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} p_i(t)p_j(t)G(x_i(t) - x_j(t)). \tag{2.14}$$

Notice that (2.13) are canonically Hamiltonian only for the CH equation (1.5) with $b = 2$, [5, 6, 7, 8, 44].

2.3. Space and time BV estimates. In what follows, we show that the total variations of the particle solution $u^N(x,t)$ and its derivative $u_x^N(x,t)$ are bounded both in space and time. To this end, we recall the definition of the total variation of a function.

Definition 2.7. *Consider a (possibly unbounded) interval $J \subseteq \mathbb{R}$ and a function $u : J \to \mathbb{R}$. The total variation of u is defined as*

$$\mathit{Tot.Var.}\{u\} \equiv \sup\left\{\sum_{j=1}^{N} |u(x_j) - u(x_{j-1})|\right\}, \tag{2.15}$$

where the supremum is taken over all $N \geq 1$ and all $(N+1)$-tuples of points $x_j \in J$ such that $x_0 < x_1 < \cdots < x_N$. If the right-hand side (RHS) of (2.15) *is bounded, then we say that u has BV and write $u \in BV(\mathbb{R})$.*

Theorem 2.8. *Let $u^N(x,t)$ and $u_x^N(x,t)$ be functions defined in* (2.3) *and* (2.4)*, respectively. Furthermore, assume that $G(x), G'(x) \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then, both $u^N \in BV(\mathbb{R} \times \mathbb{R}_+)$ and $u_x^N \in BV(\mathbb{R} \times \mathbb{R}_+)$.*

Proof. We begin with showing that total variation $\{u^N(\cdot,t)\}$ and total variation $\{u_x^N(\cdot,t)\}$ are bounded. Indeed, from the fact that the total momentum of the particle system (2.9) is conserved and the fact that for any two functions f, g and for any constant a

$$\text{Tot.Var.}\{f+g\} \leq \text{Tot.Var.}\{f\} + \text{Tot.Var.}\{g\} \quad \text{and} \quad \text{Tot.Var.}\{f(x+a)\} \leq \text{Tot.Var.}\{f\},$$

we obtain from (2.3) and (2.4)

$$\text{Tot.Var.}\{u^N(\cdot,t)\} \leq \sum_{j=1}^{N} p_j(t)\text{Tot.Var.}\{G(x)\} = M_0\text{Tot.Var.}\{G(x)\}, \tag{2.16}$$

$$\text{Tot.Var.}\{u_x^N(\cdot,t)\} \leq \sum_{j=1}^{N} p_j(t)\text{Tot.Var.}\{G'(x)\} = M_0\text{Tot.Var.}\{G'(x)\}. \tag{2.17}$$

Since the total variation of both $G(x)$ and $G'(x)$ is bounded, we conclude that $u^N(x,t)$ and $u_x^N(x,t)$ have BVs in space.

In order to prove that $u^N(x,t)$ and $u_x^N(x,t)$ have BV with respect to t as well, it now suffices to show that u^N and u_x^N are both Lipschitz continuous in time in L^1

[2, Theorem 2.6]. To this end, we first consider expression (2.3) for $u^N(x)$ to have

$$\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\, dx \le \int_{-\infty}^{\infty} \sum_{i=1}^{N} \Big| p_i(t) G(x - x_i(t)) - p_i(s) G(x - x_i(s)) \Big|\, dx.$$

Next, we add and subtract the term $\int_{-\infty}^{\infty} \sum_{i=1}^{N} p_i(t) G(x - x_i(s))\, dx$ in the RHS of the last inequality and rewrite it as

$$\begin{aligned}\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\, dx \le & \int_{-\infty}^{\infty} \sum_{i=1}^{N} p_i(t)\, |G(x - x_i(t)) - G(x - x_i(s))|\, dx \\ & + \int_{-\infty}^{\infty} \sum_{i=1}^{N} |G(x - x_i(s))| |p_i(t) - p_i(s)|\, dx.\end{aligned}$$

Using the results from [2, Lemma 2.3] and the fact that $G \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, we thus have

$$\begin{aligned}\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\, dx \le & \text{Tot.Var.}\{G(x)\} \sum_{i=1}^{N} p_i(t) |x_i(t) - x_i(s)| \\ & + ||G||_{L^1} \sum_{i=1}^{N} |p_i(t) - p_i(s)|.\end{aligned} \tag{2.18}$$

The sums in the RHS of (2.18) can now be estimated using the ODE system (2.2) as follows:

$$\begin{aligned}|x_i(t) - x_i(s)| &= \left| \int_s^t \frac{dx_i}{d\tau}\, d\tau \right| \le \int_s^t |u(x_i(\tau), \tau)|\, d\tau \le ||G||_\infty \int_s^t \sum_{j=1}^{N} p_j(\tau)\, d\tau \\ &= ||G||_\infty \sum_{j=1}^{N} p_j(0) |t - s| = ||G||_\infty M_0 |t - s|\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}|p_i(t) - p_i(s)| = \left| \int_s^t \frac{dp_i}{d\tau}\, d\tau \right| & \le (b-1) ||G'||_\infty \int_s^t p_i(\tau) \sum_{j=1}^{N} p_j(\tau)\, d\tau \\ & \le (b-1) ||G'||_\infty \int_s^t p_i(\tau)\, d\tau \sum_{j=1}^{N} p_j(0) = (b-1) ||G'||_\infty M_0 \int_s^t p_i\, d\tau.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{i=1}^{N} |p_i(t) - p_i(s)| & \le (b-1) ||G'||_\infty M_0 \int_s^t \sum_{i=1}^{N} p_i(\tau)\, d\tau \\ & = (b-1) ||G'||_\infty M_0^2 |t - s|.\end{aligned} \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18) yields

$$\begin{aligned}&\int_{-\infty}^{\infty} |u^N(x,t) - u^N(x,s)|\, dx \\ &\qquad \le \left(\text{Tot.Var.}\{G(x)\} ||G||_\infty + (b-1) ||G'||_\infty ||G||_{L^1} \right) M_0^2 |t - s|,\end{aligned}$$

proving that u^N is Lipshitz continuous in time in L^1 and thus $u^N \in BV(\mathbb{R} \times \mathbb{R}_+)$ [2, Theorem 2.6].

Similarly, from (2.4) we have

(2.21)
$$\int_{-\infty}^{\infty} |u_x^N(x,t) - u_x^N(x,s)|\, dx \le \int_{-\infty}^{\infty} \sum_{i=1}^{N} p_i(t)\, |G'(x - x_i(t)) - G'(x - x_i(s))|\, dx + \int_{-\infty}^{\infty} \sum_{i=1}^{N} |G'(x - x_i(s))||p_i(t) - p_i(s)|\, dx.$$

Substituting (2.19) and (2.20) into (2.21) and using the fact that $G' \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, we finally conclude that

$$\int_{-\infty}^{\infty} |u_x^N(x,t) - u_x^N(x,s)|\, dx \le (\text{Tot.Var.}\{G'(x)\}||G||_\infty + (b-1)||G'||_\infty ||G'||_{L^1})\, M_0^2 |t-s|,$$

which together with (2.17) proves that $u_x^N(x,t)$ is a BV function in x, t and also the statement of the theorem. ◻

3. Global weak solution and convergence analysis. In this section, we propose a new, concise method for showing the convergence of the particle solution to a unique global weak solution of the b-family of evolutionary PDEs. We restrict our attention to the specific case of the IVP (1.5), (1.2) with $b > 1$. In this case, one can explicitly compute the velocity u and its derivative by the convolutions (2.3) and (2.4), respectively, with G defined by (1.4) and G' is given by

$$G'(x) = -\frac{1}{2\alpha^2} \text{sgn}(x) e^{-|x|/\alpha}. \tag{3.1}$$

One can also easily verify that the functions G and G' defined in (1.4) and (3.1), respectively, satisfy properties (I) through (III) and calculate the total variation of G and G' explicitly:

$$\text{Tot.Var.}\{G(x)\} = 1/\alpha \quad \text{and} \quad \text{Tot.Var.}\{G'(x)\} = 2/\alpha^2. \tag{3.2}$$

We begin the section by proving for a relatively wide class of initial data there are no particle collisions in finite time and as a result there exists a unique global solution of the particle ODE system (2.2) for any $b > 1$. We then show that the particle method applied to the b-equation is a weak solution to (1.5), (1.2). Finally, we state our main convergence result, which is proved using the compactness results generated from the BV estimates established above.

3.1. Global solution of the particle system. We first prove the following important conservation law.

PROPOSITION 3.1. *Consider* (2.2)–(2.4) *for any* $b > 1$ *and* G *and* G' *given by* (1.4) *and* (3.1), *respectively, and assume that* $p_i(0) > 0$, $i = 1 \ldots N$, *and* $x_i(t) < x_{i+1}(t)$, $i = 1 \ldots N$, *at some time* t. *Then,*

$$P_N(t) = \left(\prod_{k=1}^{N} p_k(t) \right) \left(\prod_{k=1}^{N-1} \Big[G(0) - G(x_k(t) - x_{k+1}(t)) \Big]^{(b-1)} \right) \tag{3.3}$$

is constant of motion.

Proof. To establish the above proposition, it suffices to show that (see also [44, 52])

$$\frac{d}{dt}P_N(t)=0. \tag{3.4}$$

To this end, we calculate the derivative of $P_N(t)$ and write it in the following form:

$$\frac{d}{dt}P_N(t)=P_N(t)\sum_{k=1}^{N-1}\frac{(b-1)G'(x_k(t)-x_{k+1}(t))(\dot{x}_{k+1}(t)-\dot{x}_k(t))}{G(0)-G(x_k(t)-x_{k+1}(t))}+P_N(t)\sum_{k=1}^{N}\frac{\dot{p}_k(t)}{p_k(t)},$$

where $\dot{x}_k(t)$ and $\dot{p}_k(t)$ denote the derivatives of $x_k(t)$ and $p_k(t)$ with respect to time, respectively. Substituting the expressions for $\dot{x}_k(t))$ and $\dot{p}_k(t)$ from (2.2) and expressions for G and G' from (1.4) and (3.1) into the above equation, we obtain the following:

$$\begin{aligned}\frac{d}{dt}P_N(t)&=\frac{b-1}{2\alpha^2}P_N(t)\\&\times\sum_{k=1}^{N-1}\sum_{i=1}^{N}\frac{e^{(x_k(t)-x_{k+1}(t))/\alpha}\left(p_i(t)e^{-|x_{k+1}(t)-x_i(t)|/\alpha}-p_i(t)e^{-|x_k(t)-x_i(t)|/\alpha}\right)}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&+\frac{b-1}{2\alpha^2}P_N(t)\sum_{k=1}^{N}\sum_{i=1}^{N}p_i(t)\mathrm{sgn}(x_k(t)-x_i(t))e^{-|x_k(t)-x_i(t)|/\alpha}.\end{aligned} \tag{3.5}$$

By splitting up the summation terms in (3.5) into the intervals $i<k$, $i=k$, and $i>k$, the first sum becomes

$$\begin{aligned}&\sum_{k=1}^{N-1}\sum_{i=1}^{N}\frac{e^{(x_k(t)-x_{k+1}(t))/\alpha}p_i(t)\left(e^{-|x_{k+1}(t)-x_i(t)|/\alpha}-e^{-|x_k(t)-x_i(t)|/\alpha}\right)}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad=\sum_{k=1}^{N-1}\sum_{i<k}\frac{p_i(t)\left(e^{(x_k(t)+x_i(t)-2x_{k+1}(t))/\alpha}-e^{(x_i(t)-x_{k+1}(t))/\alpha}\right)}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad+\sum_{k=1}^{N-1}\frac{e^{(x_k(t)-x_{k+1}(t))/\alpha}p_k(t)\left(e^{(x_k(t)-x_{k+1}(t))/\alpha}-1\right)}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad+\sum_{k=1}^{N-1}\sum_{i>k}\frac{p_i(t)\left(e^{(x_k(t)-x_i(t))/\alpha}-e^{(2x_k(t)-x_i(t)-x_{k+1}(t))/\alpha}\right)}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad=\sum_{k=1}^{N-1}\sum_{i<k}\frac{p_i(t)(e^{(x_k(t)-x_{k+1}(t))/\alpha}-1)e^{(x_i(t)-x_{k+1}(t))/\alpha}}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad-\sum_{k=1}^{N-1}p_k(t)e^{(x_k(t)-x_{k+1}(t))/\alpha}\\&\quad+\sum_{k=1}^{N-1}\sum_{i>k}\frac{p_i(t)(1-e^{(x_k(t)-x_{k+1}(t))/\alpha})e^{(x_k(t)-x_i(t))/\alpha}}{1-e^{(x_k(t)-x_{k+1}(t))/\alpha}}\\&\quad-\sum_{k=1}^{N-1}\sum_{i<k}p_i(t)e^{(x_i(t)-x_{k+1}(t))/\alpha}-\sum_{k=1}^{N-1}p_k(t)e^{(x_k(t)-x_{k+1}(t))/\alpha}\\&\quad+\sum_{k=1}^{N-1}\sum_{i>k}p_i(t)e^{(x_k(t)-x_i(t))/\alpha}.\end{aligned} \tag{3.6}$$

Using properties of the signum function, we also split the second summation term in (3.5) into the intervals $i < k$, $i = k$, and $i > k$ to obtain

$$\begin{aligned}\sum_{k=1}^{N}\sum_{i=1}^{N} p_i(t)\mathrm{sgn}(x_k(t)-x_i(t))e^{-|x_k(t)-x_i(t)|/\alpha} \\ = \sum_{k=1}^{N}\sum_{i<k} p_i(t)e^{(x_i(t)-x_k(t))/\alpha} - \sum_{k=1}^{N-1}\sum_{i>k} p_i(t)e^{(x_k(t)-x_i(t))/\alpha}.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) and using the fact that

$$\begin{aligned}\sum_{k=1}^{N}\left(\sum_{i<k} p_i(t)e^{(x_i(t)-x_k(t))/\alpha}\right) - \sum_{k=1}^{N-1}\left(\sum_{i<k} p_i(t)e^{(x_i(t)-x_{k+1}(t))/\alpha}\right) \\ = \sum_{k=1}^{N-1} p_k(t)e^{(x_k(t)-x_{k+1}(t))/\alpha},\end{aligned}$$

the derivative in (3.5) simplifies to

$$\begin{aligned}\frac{d}{dt}P_N(t) = \frac{b-1}{2\alpha^2}P_N(t)\Bigg(&\sum_{k=1}^{N-1}\sum_{i>k} p_i(t)e^{(x_k(t)-x_i(t))/\alpha} \\ &- \sum_{k=1}^{N-1}\sum_{i<k} p_i(t)e^{(x_i(t)-x_{k+1}(t))/\alpha} - \sum_{k=1}^{N-1} p_k(t)e^{(x_k(t)-x_{k+1}(t))/\alpha}\Bigg) \\ + \frac{b-1}{2\alpha^2}P_N(t)&\left(\sum_{k=1}^{N}\sum_{i<k} p_i(t)e^{(x_i(t)-x_k(t))/\alpha} - \sum_{k=1}^{N-1}\sum_{i>k} p_i(t)e^{(x_k(t)-x_i(t))/\alpha}\right) \\ =0,&\end{aligned}$$

which establishes the proposition. ∎

Using Propositions 2.3 through 2.5 and 3.1, we can now show that for a class of initial data, particles cannot cross, and thus we establish global existence of the solution to the ODE system given by (2.2).

LEMMA 3.2. *Consider the system* (2.2) *with initial data* $p_i(0) > 0$ *and* $x_i(0) < x_{i+1}(0)$ *for any* $i = 1 \dots N$. *Then for all* $t > 0$, $x_i(t) \neq x_{i+1}(t)$ *for any* $i = 1, \dots N$ *and for all.*

Proof. Suppose on the contrary that there exist time $t^* > 0$ and number k such that

$$\lim_{t\to t^*} x_k(t) - x_{k+1}(t) = 0. \tag{3.8}$$

Then, using the fact that $P_N(0) > 0$ by our choice of initial data, we have

$$\lim_{t\to t^*} \prod_{i=1}^{N} p_i(t) = \infty, \tag{3.9}$$

which contradicts the conservation property (2.9). Hence no two particles may cross in finite time. ∎

We finally present the following global existence result for (2.2). The proof follows directly from Propositions 2.3 through 2.5, 3.1 and Lemma 3.2).

THEOREM 3.3. *If the initial momenta in the system* (2.2) *are positive, i.e.,* $p_i(0) > 0$ *and* $x_i(0) < x_{i+1}(0)$ *for any* $i = 1 \dots N$*, then the solution of system* (2.2) *exists uniquely for all* $t \in (0, \infty)$.

Remark 3.4. We do have a proof for an arbitrary (nonsmooth) kernel $G(x)$ satisfying properties (I)–(III). However, if G is smooth, the existence and uniqueness of a global solution to system (2.2) follows from the standard ODE theory.

Remark 3.5. We also note that similar results have been established in [8] and [52] for the special cases of the CH equation ((1.5) with $b = 2$) and the DP equation ((1.5) with $b = 3$), respectively, for which (1.5) is proved to be completely integrable (see, e.g., [5, 28, 31]). The no cross property for the N-peakon solution to the CH equation was proved in [8] by the iso-spectral property associated to the Lax-pair.

3.2. Consistency of the particle method. Throughout this section, we shall assume that the initial momenta are positive and that there are no particle collisions in finite time, that is, the statement of Theorem 3.3 holds.

We begin the section with a definition of a weak solution to the IVP (1.5), (1.2) and then show that the particle solution (m^N, u^N) given by (2.1), (2.3) is indeed a weak solution to the IVP.

DEFINITION 3.6. $u(x,t) \in C(0,T; H^1(\mathbb{R})), m(x,t) = u(x,t) - \alpha^2 u_{xx}(x,t)$ *is said to be a weak solution of* (1.5), (1.2) *if*

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi(x,0) m(x,0)\, dx + \int_0^{\infty}\int_{-\infty}^{\infty} \left[\phi_t(x,t) - \alpha^2 \phi_{txx}(x,t)\right] u(x,t)\, dxdt \\ &\quad + \int_0^{\infty}\int_{-\infty}^{\infty} \left[\frac{b+1}{2}\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)\right] u^2(x,t)\, dxdt \\ &\quad - \int_0^{\infty}\int_{-\infty}^{\infty} \frac{\alpha^2(b-1)}{2}\phi_x(x,t) u_x^2(x,t)\, dxdt = 0 \end{aligned} \tag{3.10}$$

for all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$.

THEOREM 3.7. *Assume that* $m_0 \in \mathcal{M}(\mathbb{R})$*; then the particle solution* $(m^N(x,t), u^N(x,t))$ *given by* (2.1), (2.3) *is a weak solution of the problem* (1.1), (1.2).

Proof. Let $m^N(x,0), m^N(x,t)$ and $u^N(x,t), u_x^N(x,t)$ be given by formulae (2.6), (2.1) and (2.3), (2.4), respectively, and let $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$ be a test function. Then the following relations are true for any ϕ:

$$\langle m^N, \phi_t\rangle = \langle u^N, \phi_t - \alpha^2 \phi_{txx}\rangle, \tag{3.11}$$

$$\langle m^N u^N, \phi_x\rangle = \left\langle (u^N)^2, \phi_x - \frac{\alpha^2}{2}\phi_{xxx}\right\rangle + \alpha^2 \left\langle (u_x^N)^2, \phi_x\right\rangle, \tag{3.12}$$

$$\left\langle m^N u_x^N, \phi\right\rangle = \left\langle \frac{\alpha^2 (u_x^N)^2 - (u^N)^2}{2}, \phi_x\right\rangle, \tag{3.13}$$

where $\langle \cdot,\cdot\rangle$ denotes a scalar product in $\mathbb{R}\times\mathbb{R}_+$, i.e., $\langle m^N, \phi_t\rangle = \int_0^\infty \int_{-\infty}^\infty m^N(x,t)\phi_t(x,t)\, dxdt$, etc. See Proposition A.3 for a complete proof of the above relations.

Using (3.11)–(3.13) and substituting $m^N(x,t)$ as defined by (2.1) into (3.10) yields

$$\begin{aligned}\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) &+ \int_0^\infty \sum_{i=1}^{N} p_i(t)\phi_t\,(x_i(t),t)\,dt \\ &+ \int_0^\infty \sum_{i=1}^{N} p_i(t)u^N(x_i(t),t)\phi_x\,(x_i(t),t)\,dt \\ &- (b-1)\int_0^\infty \sum_{i=1}^{N} p_i(t)u_x^N\,(x_i(t),t)\,\phi\,(x_i(t),t)\,dt = 0.\end{aligned} \tag{3.14}$$

We now add and subtract $\sum_{i=1}^N \int_0^\infty p_i(t)\,\frac{dx_i}{dt}\phi_x(x_i(t),t)\,dt$ into the last equation, use the fact that

$$\frac{d\phi(x_i(t),t)}{dt} = \phi_x(x_i(t),t)\frac{dx_i(t)}{dt} + \phi_t(x_i(t),t),$$

and rewrite (3.14) as follows:

$$\begin{aligned}\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) &+ \int_0^\infty \sum_{i=1}^{N} p_i(t)\frac{d\phi(x_i(t),t)}{dt}\,dt \\ &\int_0^\infty \sum_{i=1}^{N} p_i(t)\left[u^N(x_i(t),t) - \frac{dx_i(t)}{dt}\right]\phi_x\,(x_i(t),t)\;dt \\ &- (b-1)\int_0^\infty \sum_{i=1}^{N} p_i(t)u_x^N\,(x_i(t),t)\,\phi\,(x_i(t),t)\;dt = 0.\end{aligned} \tag{3.15}$$

Integrating by parts the second term in the first row in (3.15), and rearranging other terms, we finally obtain

$$\begin{aligned}&\int_0^\infty \sum_{i=1}^{N} p_i(t)\left[\frac{dx_i(t)}{dt} - u^N(x_i(t),t)\right]\phi_x\,(x_i(t),t)\;dt \\ &+ \int_0^\infty \sum_{i=1}^{N} \left[\frac{dp_i(t)}{dt} + (b-1)p_i(t)u_x^N\,(x_i(t),t)\right]\phi(x_i(t),t)\,dt = 0.\end{aligned} \tag{3.16}$$

Since the functions $x_i(t)$ and $p_i(t)$ satisfy the system (2.2), the last equation holds for any ϕ implying that m^N, u^N defined by (2.1), (2.3) is a weak solution of (1.1), (1.2). This completes the proof. ∎

3.3. Compactness and convergence. We are now in a position to establish a convergence result for the particle method applied to (1.1). Using the BV estimates for $u^N(x,t)$ and $u_x^N(x,t)$, and the fact that the particle solution satisfies the equation in the sense of distributions, we may establish the following convergence result, which in turn proves the existence of a unique global weak solution to the b-equation (1.1) with any $b > 1$. Once again, we assume that the statement of Theorem 3.3 holds.

THEOREM 3.8. *Suppose that* $(m^N(x,t), u^N(x,t))$ *is a particle solution of* (2.1), (2.2) *with initial approximation* $m^N(\cdot,0) \stackrel{*}{\rightharpoonup} m_0$, $m_0 \in \mathcal{M}_+(\mathbb{R})$ *with compact support and bounded mass* $|m_0|$. *Then, for any* $T > 0$ *there exists a unique global weak* *solution* (u, m) *of* (1.5), (1.2) *for any* $b > 1$ *with the regularity* $m \in \mathcal{M}(\mathbb{R} \times (0, T))$, $u, u_x \in BV(\mathbb{R} \times (0, T))$ *and* $u \in Lip(0, T; H^1(\mathbb{R}))$.

Proof. Using BV estimates for $u^N(x,t)$ and $u_x^N(x,t)$, we refer to the compactness property in [2, Theorem 2.4] and conclude that there exist functions $u(x,t)$ and $u_x(x,t)$ and a subsequence (still labeled as $u^N(x,t)$) such that

$$\lim_{N\to 0} ||u^N - u||_{L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)} = 0, \qquad \lim_{N\to 0} ||u_x^N - u_x||_{L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)} = 0. \tag{3.17}$$

From Proposition 3.7, we know that the particle solution (m^N, u^N) is a weak solution of (1.1) and thus satisfy

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(x,0) m^N(x,0)\,dx &+ \int_0^{\infty}\int_{-\infty}^{\infty} \left[\phi_t(x,t) - \alpha^2\phi_{txx}(x,t)\right] u^N(x,t)\,dxdt \\ &+ \int_0^{\infty}\int_{-\infty}^{\infty} \left[\frac{b+1}{2}\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)\right](u^N)^2(x,t)\,dxdt \\ &+ \int_0^{\infty}\int_{-\infty}^{\infty} \frac{\alpha^2(b-1)}{2}\phi_x(x,t)(u_x^N)^2(x,t)\,dxdt = 0.\end{aligned} \tag{3.18}$$

To complete the proof, we need to show that each term in (3.18) converges to that of the limit solution (m, u) in (3.10).

Indeed, by the construction of the initial approximation and Proposition 2.2, we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \phi(x,0) m^N(x,0)\,dx = \int_{-\infty}^{\infty} \phi(x,0) m(x,0)\,dx. \tag{3.19}$$

Furthermore, from (3.17) and the fact that $u^N \in \mathrm{BV}(\mathbb{R}\times\mathbb{R}_+)$ and $u_x^N \in \mathrm{BV}(\mathbb{R}\times\mathbb{R}_+)$ follows that

$$\begin{aligned}&\left|\int_0^{\infty}\int_{-\infty}^{\infty} \left(u^N(x,t)^2 - u(x,t)^2\right)\phi(x,t)\,dxdt\right| \\ &\quad = \Big|\int_0^{\infty}\int_{-\infty}^{\infty} (u^N(x,t) + u(x,t))(u^N(x,t) - u(x,t))\phi(x,t)\,dxdt\Big| \\ &\quad \le \|\phi\|_{L^\infty}(\|u^N\|_{L^\infty} + \|u\|_{L^\infty}) \iint_{(x,t)\in \mathrm{supp}\{\phi\}} \left|u^N(x,t) - u(x,t)\right| dxdt\end{aligned}$$

holds for any $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, and thus

$$\left\langle (u^N)^2, \phi\right\rangle \to \left\langle u^2, \phi\right\rangle \quad \text{as} \quad N \to \infty. \tag{3.20}$$

Similarly, for any $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ we have

$$\left\langle (u_x^N)^2, \phi\right\rangle \to \left\langle u_x^2, \phi\right\rangle \quad \text{as} \quad N \to \infty, \tag{3.21}$$

and therefore

$$\langle u^N, \phi_t - \alpha^2\phi_{txx}\rangle \to \langle u, \phi_t - \alpha^2\phi_{txx}\rangle \tag{3.22}$$

$$\left\langle (u^N)^2, \phi_x - \frac{\alpha^2}{2}\phi_{xxx}\right\rangle \to \left\langle (u)^2, \phi_x - \frac{\alpha^2}{2}\phi_{xxx}\right\rangle, \tag{3.23}$$

$$\left\langle \frac{\alpha^2(u_x^N)^2 - (u^N)^2}{2}, \phi_x\right\rangle \to \left\langle \frac{\alpha^2(u_x)^2 - (u)^2}{2}, \phi_x\right\rangle \tag{3.24}$$

as $N \to \infty$. This shows that the limit (m, u) is indeed a weak solution to the b-equation (1.1).

It should be observed that since $G, G' \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, hence $G, G' \in L^2(\mathbb{R})$ and thus with the bounds (2.9), we have $u \in L^\infty(0, T; H^1(\mathbb{R}))$ and $u, u_x \in \mathrm{BV}(\mathbb{R} \times \mathbb{R}_+)$. The latter implies that (see, e.g., [2])

$$\int_{\mathbb{R}} |u(x,t) - u(x,s)|\, dx \le C_1 |t - s|, \qquad \int_{\mathbb{R}} |u_x(x,t) - u_x(x,s)|\, dx \le C_2 |t - s|$$

and thus

$$\begin{aligned} ||u(\cdot,t)||_{H^1}^2 - ||u(\cdot,s)||_{H^1}^2 \le & \int_{\mathbb{R}} |u(x,t) - u(x,s)|^2 + |u_x(x,t) - u_x(x,s)|^2\, dx \\ \le & \, 2||u||_\infty \int_{\mathbb{R}} |u(x,t) - u(x,s)|\, dx \\ & + 2||u_x||_\infty \int_{\mathbb{R}} |u_x(x,t) - u_x(x,s)|\, dx \le C|t - s|, \end{aligned}$$

proving that $u \in C(0, T; H^1(\mathbb{R}))$.

Finally, we remark that the weak solution for the b-family equation (1.5), (1.2) is unique in the obtained class of functions. The result has been proved in [25] for the CH equation ($b = 2$), by direct estimations for the equation recast in the conservative form

$$u_t + u u_x + G' * \left[u^2 + \frac{1}{2} u_x^2 \right] = 0, \tag{3.25}$$

where G' is given by (3.1), as before. The proof of the uniqueness result for any $b > 1$ follows directly from [25] by rewriting (1.5) as

$$u_t + u u_x + G' * \left[\frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 \right] = 0. \qquad \square \tag{3.26}$$

Remark 3.9. We also note that for the special case of the CH equation ($b = 2$), the convergence of the particle method to a smooth solution has been verified in [7, 8].

4. Conclusion. In this paper, the concept of BV functions was used to establish the convergence of the particle method applied to the b-equation (1.1) for a special choice of the convolution kernel G and under a suitable class of initial data. These BV estimates were derived by using conservation properties associated with the particle system. In turn, our convergence results allowed us to provide a novel method for proving the existence of a unique global weak solution to (1.1) for G given by (1.4) and for any $b > 1$.

To this extent, we have only provided a theoretical study of the convergence of the particle method applied to (1.5). In the future, numerical experiments will be conducted that illustrate the performance of the particle method applied to the b-equations. In particular, we shall focus on the cases where the resulting PDE is completely integrable ($b = 2, 3$) and hence will yield peakons as their solutions. We will also analyze the convergence of the particle method applied to the analogous 2-D version of (1.5) more commonly referred to as the EPDiff equation and perform numerical experiments with arbitrary initial data.

Appendix A. This appendix provides additional details and proofs of propositions omitted above.

Proposition A.1. *Suppose that $G(x)$ and $G'(x)$ are given by* (1.4) *and* (3.1), *respectively. Then the following relation is true for any $\phi(x) \in C_0^\infty(\mathbb{R})$:*

$$G(x_1 - x_2)\,(\phi'(x_1) + \phi'(x_2)) = 2\int_{-\infty}^{\infty} G(x - x_1)G(x - x_2)\left(\phi'(x) - \frac{\alpha^2}{2}\phi'''(x)\right) dx + 2\alpha^2\int_{-\infty}^{\infty} G'(x - x_1)G'(x - x_2)\phi'(x)\,dx. \tag{A.1}$$

Proof. We consider both the cases where $x_1 = x_2$ and $x_1 < x_2$. If $x_1 = x_2$, then (A.1) reduces to the following:

$$\frac{1}{\alpha}G(0)\phi'(x_1) = 2\int_{-\infty}^{\infty} G^2(x - x_1)\left(\phi'(x) - \frac{\alpha^2}{2}\phi'''(x)\right) dx + 2\alpha^2\int_{-\infty}^{\infty}(G'(x - x_1))^2\phi'(x)\,dx.$$

Splitting the above integrals into two regions ($x < x_1$ and $x > x_1$), integrating the term containing $\phi'''(x)$ twice, and combining like terms proves the equality.

We now consider the case where $x_1 < x_2$. We split the integrals into three regions ($x < x_1$, $x_1 < x < x_2$, and $x > x_2$) and integrate the term containing $\phi'''(x)$ by parts twice to obtain

$$\begin{aligned} -\alpha^2\int_{-\infty}^{\infty} G(x - x_1)G(x - x_2)\phi'''(x)\,dx &= \frac{1}{2\alpha}e^{(x_1 - x_2/\alpha)}\phi'(x_1) - \frac{1}{\alpha^2}\int_{-\infty}^{x_1} e^{(x - x_1)/\alpha + (x - x_2)/\alpha}\phi'(x)\,dx \\ &\quad + \frac{1}{2\alpha}e^{(x_1 - x_2)/\alpha}\phi'(x_2) - \frac{1}{\alpha^2}\int_{x_2}^{\infty} e^{(x_1 - x)/\alpha + (x_2 - x)/\alpha}\phi'(x)\,dx. \end{aligned} \tag{A.2}$$

We also have the following:

$$\begin{aligned} 2\int_{-\infty}^{\infty} G(x - x_1)G(x - x_2)\phi'(x)\,dx &= \frac{1}{2\alpha^2}\int_{-\infty}^{x_1} e^{-(x_1 - x)/\alpha - (x_2 - x)/\alpha}\phi'(x)\,dx \\ &\quad + \frac{1}{2\alpha^2}\int_{x_1}^{x_2} e^{(x_1 - x)/\alpha - (x_2 - x)/\alpha}\phi'(x)\,dx \\ &\quad + \frac{1}{2\alpha^2}\int_{x_2}^{\infty} e^{-(x - x_1)/\alpha - (x - x_2)/\alpha}\phi'(x)\,dx \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} 2\alpha^2\int_{-\infty}^{\infty} G'(x - x_1)G'(x - x_2)\phi'(x)\,dx &= \frac{1}{2\alpha^2}\int_{-\infty}^{x_1} e^{-(x_1 - x)/\alpha - (x_2 - x)/\alpha}\phi'(x)\,dx \\ &\quad + \frac{1}{2\alpha^2}\int_{x_1}^{x_2} e^{(x_1 - x)/\alpha - (x_2 - x)/\alpha}\phi'\,dx \\ &\quad + \frac{1}{2\alpha^2}\int_{x_2}^{\infty} e^{-(x - x_1)/\alpha - (x - x_2)/\alpha}\phi'(x)\,dx. \end{aligned} \tag{A.4}$$

Combining (A.2), (A.3), and (A.4), we obtain

$$\frac{1}{2\alpha}e^{(x_1-x_2)/\alpha}\phi'(x_1)+\frac{1}{2\alpha}e^{(x_1-x_2)/\alpha}\phi'(x_2)=G(x_1-x_2)\left(\phi'(x_1)+\phi'(x_2)\right),$$

and hence the proposition is proved. ∎

Proposition A.2. *Suppose that $G(x)$ and $G'(x)$ are given by* (1.4) *and* (3.1), *respectively. Then the following relation is true for any $\phi(x)\in C_0^\infty(\mathbb{R})$:*

(A.5)
$$G'(x_1-x_2)(\phi(x_1)-\phi(x_2)) = \int_{-\infty}^{\infty}\left[\alpha^2 G'(x-x_1)G'(x-x_2)-G(x-x_1)G(x-x_2)\right]\phi'(x)\,dx.$$

Proof. As before, we first consider the case where $x_1=x_2$. Then the problem reduces to showing

$$\int_{-\infty}^{\infty}\left[\alpha^2\left(G'(x-x_1)\right)^2-\left(G(x-x_1)\right)^2\right]\phi'(x)\,dx=0. \tag{A.6}$$

Indeed, by definition of $G(x)$ and its derivative in (1.4) and (3.1), we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[\alpha^2\left(G'(x-x_1)\right)^2-\left(G(x-x_1)\right)^2\right]\phi'(x)\,dx\\ &\quad=\frac{1}{4\alpha^2}\int_{-\infty}^{\infty}\left[e^{-2|x-x_1|/\alpha}-e^{-2|x-x_1|/\alpha}\right]\phi'(x)\,dx=0.\end{aligned}$$

We now consider the case where $x_1<x_2$ and split the integrals as follows:

(A.7)
$$\begin{aligned}-\int_{-\infty}^{\infty}G(x-x_1)G(x-x_2)\phi'(x)\,dx=&-\frac{1}{4\alpha^2}\int_{-\infty}^{x_1}e^{-(x_1-x)/\alpha-(x_2-x)/\alpha}\phi'(x)\,dx\\ &+\frac{1}{4\alpha^2}\int_{x_1}^{x_2}e^{-(x-x_1)/\alpha-(x_2-x)/\alpha}\phi'(x)\,dx\\ &+\frac{1}{4\alpha^2}\int_{x_2}^{\infty}e^{-(x-x_1)/\alpha-(x-x_2)/\alpha}\phi'(x)\,dx\end{aligned}$$

and

(A.8)
$$\begin{aligned}\alpha^2\int_{-\infty}^{\infty}G'(x-x_1)G'(x-x_2)\phi'(x)\,dx=&\frac{1}{4\alpha^2}\int_{-\infty}^{x_1}e^{-(x_1-x)/\alpha-(x_2-x)/\alpha}\phi'(x)\,dx\\ &-\frac{1}{4\alpha^2}\int_{x_1}^{x_2}e^{-(x-x_1)/\alpha-(x_2-x)/\alpha}\phi'(x)\,dx\\ &+\frac{1}{4\alpha^2}\int_{x_2}^{\infty}e^{-(x-x_1)/\alpha-(x-x_2)/\alpha}\phi'(x)\,dx.\end{aligned}$$

By combining (A.7) and (A.8) and integrating once, we obtain

$$-\frac{1}{2\alpha^2}e^{-(x_2-x_1)/\alpha}\left(\phi(x_2)-\phi(x_1)\right)=G'(x_1-x_2)\left(\phi(x_1)-\phi(x_2)\right).$$

This proves the proposition. ∎

PROPOSITION A.3. *Suppose that $m^N(x,t)$, $u^N(x,t)$, and $u_x^N(x,t)$ are given by* (2.1), (2.3), *and* (2.4), *respectively. Then relations* (3.11)–(3.13) *are true for any* $\phi(x,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$.

Proof. To begin, we first prove the relation (3.11), which implies that $m^N(x,t) = u^N(x,t) - \alpha^2 u_{xx}^N(x,t)$ in the sense of distributions. Indeed, for any $\phi(x,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, we have the following relation by direct substitution of (2.1) into the left-hand side of (3.11) and integration by parts:

$$\left\langle u^N - \alpha^2 u_{xx}^N, \phi_t \right\rangle = \left\langle u^N, \phi_t \right\rangle + \alpha^2 \left\langle u_x^N, \phi_{tx} \right\rangle.$$

Using (2.3) and (2.4) and integrating by parts once again, we prove (3.11):

$$\begin{aligned}
\left\langle u^N - \alpha^2 u_{xx}^N, \phi_t \right\rangle &= \int_0^\infty \sum_{i=1}^N p_i(t) \int_{-\infty}^\infty G(x-x_i(t))\phi_t(x,t)\,dxdt \\
&\quad + \alpha^2 \int_0^\infty \sum_{i=1}^N p_i(t) \int_{-\infty}^\infty G'(x-x_i(t))\phi_{tx}(x,t)\,dxdt \\
&= \int_0^\infty \sum_{i=1}^N p_i(t) \int_{-\infty}^\infty G(x-x_i(t))\left(\phi_t(x,t) - \alpha^2\phi_{txx}(x,t)\right)\,dxdt \\
&= \langle u^N, \phi_t - \alpha^2\phi_{txx}\rangle.
\end{aligned}$$

Next, we verify (3.12) as follows. Direct substitution shows that

$$\left\langle m^N u^N, \phi_x \right\rangle = \int_0^\infty \sum_{i=1}^N\sum_{j=1}^N p_i(t)p_j(t)G(x_i(t)-x_j(t))\phi_x(x_i(t),t)\,dt.$$

Using Proposition A.1 and the fact that $G(x)$ is an even function, we find that

$$\begin{aligned}
&\left\langle m^N u^N, \phi_x \right\rangle \\
&= \frac{1}{2}\int_0^\infty \sum_{i=1}^N\sum_{j=1}^N p_i(t)p_j(t)G(x_i(t)-x_j(t))\left(\phi_x(x_i(t),t) + \phi_x(x_j(t),t)\right)\,dt \\
&= \int_0^\infty \sum_{i=1}^N\sum_{j=1}^N p_i(t)p_j(t)\Bigg[\int_{-\infty}^\infty G(x-x_i(t))G(x-x_j(t)) \\
&\qquad\qquad \left(\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)\right)\,dx \\
&\qquad\qquad + \ \alpha^2\int_{-\infty}^\infty G'(x-x_i(t))G'(x-x_j(t))\phi_x(x,t)dx\Bigg]dt \\
&= \left\langle (u^N)^2, \phi_x - \frac{\alpha^2}{2}\phi_{xxx}\right\rangle + \alpha^2\left\langle (u_x^N)^2, \phi_x\right\rangle.
\end{aligned}$$

Finally, in order to prove (3.13), we proceed similarly by first observing that

$$\left\langle m^N u_x^N, \phi \right\rangle = \int_0^\infty \sum_{i=1}^N\sum_{j=1}^N p_i(t)p_j(t)G'(x_i(t)-x_j(t))\phi(x_i(t),t)\,dt.$$

We use Proposition A.2 and the fact that $G'(x)$ is an odd function to obtain

$$\begin{aligned}
&\langle m^N u_x^N, \phi\rangle \\
&\quad = \frac{1}{2}\int_0^\infty \sum_{i=1}^N \sum_{j=1}^N p_i(t)p_j(t)G'(x_i(t)-x_j(t))\left(\phi(x_i(t),t)-\phi(x_j(t),t)\right)dt \\
&\quad = \frac{1}{2}\int_0^\infty \sum_{i=1}^N \sum_{j=1}^N p_i(t)p_j(t)\Big[-\int_{-\infty}^\infty G(x-x_i(t))G(x-x_j(t))\phi_x(x,t)\,dx \\
&\qquad\qquad + \frac{\alpha^2}{2}\int_{-\infty}^\infty G'(x-x_i(t))G'(x-x_j(t))\phi_x(x,t)\,dx\Big]\,dt \\
&\quad = \left\langle \frac{\alpha^2(u_x^N)^2-(u^N)^2}{2}, \phi_x\right\rangle. \qquad \square
\end{aligned}$$

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