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On the stability of a class of self-similar solutions to the filtration-absorption equation

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We consider the one-dimensional and two-dimensional filtration-absorption equation $u_t = u\Delta u-(c-1)(\nabla u)^2$. The one-dimensional case was considered previously by Barenblatt *et al.* [4], where a special class of self-similar solutions was introduced. By the analogy with the 1D case we construct a family of axisymmetric solutions in 2D. We demonstrate numerically that the self-similar solutions obtained attract the solutions of non-self-similar Cauchy problems having the initial condition of compact support. The main analytical result we provide is the linear stability of the above self-similar solutions both in the 1D case and in the 2D case.

1 Introduction

We consider the filtration-absorption equation

$$u_t = u\Delta u - (c-1)(\nabla u)^2 = \nabla\,(u\nabla u) - c(\nabla u)^2 \tag{1.1 a}$$

in $\mathbb{R}^d \times \mathbb{R}^+$ subject to initial condition

$$u(x,t) = u_0(x). \tag{1.1 b}$$

Here d is either one or two, $c > 1$ is a constant and $u_0(x)$ is a bounded continuous nonnegative function on $\mathbb{R}^d$ with compact support. According to numerous authors [8, 10, 12, 13, 15], this type of equation arises in several biological and physical contexts.

It was shown [4] that equation (1.1 *a*) can be derived as a model of the groundwater flow in water-absorbing *fissurized porous rock*. For a purely porous medium the absorption coefficient c is always less than one, because the absorption can not exceed the available amount of water. However, if the rock is fissurized, c can be substantially larger than one (see Barenblatt *et al.* [4] for detailed derivation of the equation). In Barenblatt *et al.* [4], the one-dimensional case was considered and a special family of self-similar solutions of (1.1 *a, b*) with shrinking support was constructed and investigated analytically and numerically. Since in the subsequent text we discuss stability properties of these solutions and their generalizations, we would like to give here a brief account of these self-similar solutions. For sufficiently large c (in fact $c > 3/2$) equation (1.1 *a*) has self-similar solutions vanishing in finite time t_0 at a certain point x_0

$$u = B^2\mu\,(t_0-t)^{2\mu-1}\,F\left(\frac{x-x_0}{x_f}\right), \qquad x_f = B\,(t_0-t)^{\mu}\,. \tag{1.2}$$

Here x_f is the contracting half-width of the support, B and μ are constants. It has been assumed in Barenblatt *et al.* [4] that the structure of the groundwater 'dome' is symmetric, so that the function $F(\xi)$, $\xi = \dfrac{x - x_0}{x_f}$ is an even function determined in the interval $-1 \leqslant \xi \leqslant 1$ and $F(\xi)$ vanishes for $|\xi| > 1$. Solution (1.2) is a self-similar solution of the second kind (see Barenblatt *et al.* [3]): conservation laws and dimensional considerations are insufficient to obtain the exponent μ. To determine the exponent μ and the function F, it is necessary to solve a nonlinear eigenvalue problem. The constants B and t_0, as well as x_0, depend on initial data of the original problem. They can be obtained (see Barenblatt *et al.* [4]) numerically by matching the self-similar solution with the solution of the Cauchy problem at the non-self-similar stage.

The nonlinear eigenvalue problem to determine F and μ is as follows:

$$F\frac{d^2F}{d\xi^2} - (c-1)\left(\frac{dF}{d\xi}\right)^2 - \xi\frac{dF}{d\xi} + \frac{2\mu - 1}{\mu}F = 0\,, \quad \xi = \frac{x - x_0}{B\,(t_0 - t)^{\mu}}, \tag{1.3a}$$

$$F'(0) = 0, \quad F(1) = 0, \quad F'(1) = -\frac{1}{c-1}. \tag{1.3b}$$

The boundary conditions (1.3 b) express the symmetry of the solution and continuity of the solution and the flux at the free boundary.

Considering a special case when $F(\xi)$ has a maximum at $\xi = 0$ the authors searched the solution in the form of an expansion

$$F(\xi) = \sum_{n=1}^{\infty} a_n \left(1 - \xi^2\right)^n .$$

Since the terms of the sum (except the first one) do not contribute to all three boundary conditions of the eigenvalue problem, the only possible solution is

$$a_1 = \frac{1}{2(c-1)}, \quad a_n = 0 \quad \forall n \geqslant 2,$$

resulting in

$$F(\xi) = \frac{1}{2(c-1)}\left(1 - \xi^2\right), \quad \xi = \frac{x - x_0}{B\,(t_0 - t)^{\mu}}, \tag{1.4}$$

and

$$\mu = \frac{c-1}{2c-3}. \tag{1.5}$$

Hence, the self-similar solution (1.2) assumes the form

$$\begin{aligned} u_s &= \frac{1}{2(2c-3)} B^2\,(t_0 - t)^{\frac{1}{2c-3}} \left[1 - \frac{(x - x_0)^2}{B^2\,(t_0 - t)^{\frac{2(c-1)}{2c-3}}}\right]_+ , \\ x_f &= B\,(t_0 - t)^{\frac{(c-1)}{2c-3}} , \end{aligned} \tag{1.6}$$

which is appropriate for $c > 3/2$, where the time of collapse t_0 is finite. Solution (1.6) is a weak, non-classical solution that is positive (the notation $x_+ \equiv \max(x, 0)$ is used) in an interval $[-x_f(t), x_f(t)]$ and that vanishes elsewhere.

The solution (1.6) has essentially different behaviour in various intervals of the values of the absorption coefficient c. For $1 < c < 3/2$, the compact support contracts as well, but the collapse time is infinite. In this case, μ becomes negative, and it is convenient to replace μ by $-\mu$ and t_0 by $-t_0$. The solution (1.2) may be represented in the form

$$u_s = \frac{1}{2(3-2c)} B^2 (t_0 + t)^{-\frac{1}{3-2c}} \left[1 - \frac{(x - x_0)^2}{B^2 (t_0 + t)^{-\frac{2(c-1)}{3-2c}}} \right]_+ ,$$
$$x_f = B (t_0 + t)^{-\frac{(c-1)}{3-2c}} . \tag{1.7}$$

Since the time of collapse is now infinite, t_0 becomes an additive constant.

For $c \to 3/2^{\pm}$ solution (1.2) tends to a finite limit, so that $u(x_0, t) = u_{\max}(t)$ and x_f decay with time according to exponential laws (see Barenblatt *et al.* [4] for more details).

It should be observed that solution (1.6) as well as (1.7) belong to a class of solutions studied by Angenent [1], where it was assumed that $u(x, t)$ can be expanded in a Taylor series of sufficiently high order at the free boundary. The large time asymptotics in the one-dimensional case when $c < 3/2$ was also considered by Angenent [2].

A detailed analytic study of the problem under consideration was conducted by Bertsch, Dal Passo and Ughi in a series of works [6]–[8]. Considering continuous, nonnegative and bounded initial data they used the classical viscosity solution method to construct a weak solution of $(1.1\,a, b)$ whose support is neither expanding nor contracting. The solution was obtained as a limit of the non-degenerate problem with initial data $u_\epsilon(x, 0) = u(x, 0) + \epsilon$ when $\epsilon \to 0$. It was proved that the 'viscosity' solution is maximal in a class of continuous weak solutions and therefore unique. In Bertsch & Dal-Passo [5] a numerical approximation of $(1.1\,a, b)$ leading to the 'viscosity' solution was proposed.

It turns out, however, that solutions of $(1.1\,a, b)$ are not uniquely determined by the initial data $u_0(x)$. First, for the case $c = 1$, Ughi [15] and independently Dal-Passo & Luckhaus [10] discovered that problem $(1.1\,a, b)$ may possess other solutions than the 'viscosity' solution. In Bertsch *et al.* [7] their results were extended to the case $c > 1$, and a general discussion of the nonuniqueness phenomenon was performed. It was shown that if $u_0(x) > 0$ inside the support than for any given $T > 0$ there exist infinitely many solutions $u(x, t)$ of $(1.1\,a, b)$ with shrinking support and T as 'extinction time', i.e. $u(x, t) \not\equiv 0$ for $t < T$ and $u(x, t) \equiv 0$ for $t \geqslant T$. If the initial data $u_0(x)$ has an isolated zero at $x = 0$, it is not clear at which time t^* a solution of $(1.1\,a, b)$ becomes positive at $x = 0$, if at all. It was shown [8] that, depending on the value of the absorption coefficient c and the local behavior of $u_0(x)$ near the origin, the 'waiting time' t^* may be zero, nonzero and finite, or infinite. In addition it was proved [7] that if $u_0(x) > 0$ in $\mathbb{R}^d \setminus \{0\}$ and $c > 1 + d/2$ nonuniqueness appears even in the class of classical solutions.

The case $c > 1 + d/2$ was also studied by Chasseigne & Vazquez [9]. They enlarged the class of initial data and considered unbounded data. Existence of a continuous weak solution was proved for every measurable initial data $u_0(x) \geqslant 0$. Uniqueness was obtained in the class of solutions whose zero level set is constant in time, and all zeros are quadratic.

In the present work, we study stability properties of the special class of self-similar solutions (1.6), (1.7) described above. From this point on we will refer to these solutions as basic or 'parabola' solutions. We start in § 2 by proving the linear stability of the basic self-similar solution. We show, however, that if one does not assume that $F(\xi)$ has a maximum at $\xi = 0$ than there exist self-similar solutions different from (1.6), (1.7). For $1 < c < 2$ we find self-similar solutions having a local minimum at $\xi = 0$ $(F(0) = F'(0) = 0)$ in an explicit form. We leave, however, an analytical investigation of these and other solutions to future work, and demonstrate *only* numerically that being perturbed by a very small perturbation at the origin the new solutions tend to the self-similar asymptotics corresponding to 'parabola' (1.6) (or (1.7)).

In § 3 we obtain a family of axisymmetric self-similar solutions, and prove the linear stability of these solutions. We investigate numerically the evolution of non-self-similar solutions of (1.1 *a*) in 2D. Numerical experiments indicate that the axisymmetric self-similar solution obtained represents intermediate asymptotics for the solutions of the Cauchy problems having an 'arbitrary' axisymmetric initial condition of compact support.

2 The one-dimensional case

2.1 Linear stability of the basic self-similar solutions

In this section we investigate the stability of the self-similar solution (1.6) (or (1.7)). We define (see Barenblatt [3]) a self-similar solution to be *stable* if the solution of any perturbed problem with a sufficiently small perturbation can be represented in the form of a self-similar solution corresponding to a constant B' which is in general different from B, plus some additional term whose ratio to the unperturbed solution tends to zero as $t \to \infty$. We shall discuss in detail the case when $c > 3/2$. The case when $c < 3/2$ can be treated in just the same way.

Since $u(x,t)$ has a discontinuous derivative $\partial_x u$ at the boundary, we shall consider the function $w_s(x,t) = u_s^2(x,t)$ (rather than $u_s(x,t)$ itself) which satisfies the equation

$$2\sqrt{w}w_t = 2ww_{xx} - c\,(w_x)^2\,, \tag{2.1}$$

and has a continuous first derivative $\partial_x w_s$. The function w_s corresponding to the basic self-similar solution is defined by

$$w_s = B^4\mu^2(t_0-t)^{4\mu-2}F^2(\xi), \quad \xi = \frac{x-x_0}{B(t_0-t)^{\mu}}, \tag{2.2}$$

for $|\xi| \leqslant 1$ and $w_s \equiv 0$ for $|\xi| \geqslant 1$. Recall that $F(\xi)$ and μ are given by (1.4) and (1.5), respectively.

Consider a perturbed solution in the form

$$w(\xi,t) = B^4\mu^2(t_0-t)^{4\mu-2}\left[F^2(\xi) + \delta^2\phi(\xi,\tau)\right], \tag{2.3}$$

where $\phi(\xi,\tau)$ is a perturbation and δ is a small parameter. (Instead of the time t it is convenient to take $\tau = -\mu\ln(t_0-t)$ as an independent variable for the perturbation.) It should be observed that, due to the perturbation, the boundary is also displaced, and we deal now with bounded perturbed support $-1-\beta_2(\tau) \leqslant \xi \leqslant 1+\beta_1(\tau)$. The perturbation is not necessarily symmetric, so $\beta_1(\tau) \neq \beta_2(\tau)$. The displacement of the boundary is proportional to the small parameter δ. Indeed, setting $\xi = 1+\beta_1(\tau)$ in (2.3), linearizing, and keeping in mind that $F^2(1) = (F^2)'(1) = 0$ and $w = 0$ for $\xi = 1+\beta_1(\tau)$, we obtain

$$\frac{\beta_1^2(\tau)}{2}(F^2)''(1) + \delta^2\phi(1,\tau) = 0, \tag{2.4}$$

whence it follows that $\beta_1(\tau)$ is proportional to δ. By setting $\xi = -1 - \beta_2(\tau)$, we obtain an analogous equation for $\beta_2(\tau.)$.

The problem is now to find how the function $\phi(\xi, \tau)$ behaves with time. Substituting (2.3) into (2.1) and linearizing we find that ϕ is given by the equation:

$$\phi_\tau = L_\xi \phi = \frac{1}{2(c-1)} \left[(1-\xi^2)\phi_{\xi\xi} + 2(c+1)\xi\phi_\xi + 4(c+1)\frac{\phi}{1-\xi^2} - 2(2c+1)\phi \right]. \quad (2.5)$$

The functions ϕ and ϕ_ξ must be both continuous at the edge of the support, namely, $\phi = \phi_\xi = 0$.

We seek a solution of (2.5) by separating the variables according to

$$\phi(\xi, \tau) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \tau} \Phi_n(\xi) = \sum_{n=0}^{\infty} a_n (t_0 - t)^{\mu \lambda_n} \Phi_n(\xi) \ , \quad (2.6)$$

where the function $\Phi_n(\xi)$ is the eigenfunction corresponding to the nth eigenvalue λ_n of the operator L_ξ. According to (2.6), if it could be shown that all the eigenvalues λ_n are non-negative and the set of eigenfunctions is complete, then the stability of the self-similar solution would be proved.

For the operator at hand, we find the whole spectrum of eigenvalues by using the fact that the eigenfunctions of this operator can be expressed in terms of Jacobi polynomials. We have

$$\lambda_n = \frac{(n+1)(n+2c)}{2(c-1)}, \quad \Phi_n(\xi) = (1-\xi^2)^{c+1} P_n^{(c,c)}(\xi). \quad (2.7)$$

Here $P_n^{(c,c)}(\xi)$ is the Jacobi polynomial of degree n defined by the recurrence formula [14]

$$P_n^{(c,c)}(\xi) = \frac{n+c}{n(n+2c)} \left[(2n+2c-1)\xi P_{n-1}^{(c,c)}(\xi) - (n+c-1) P_{n-2}^{(c,c)}(\xi) \right],$$
$$P_0^{(c,c)}(\xi) = 1, \quad P_1^{(c,c)}(\xi) = (c+1)\xi. \quad (2.8)$$

As it follows from (2.7), the set $\{\Phi_n(\xi)\}$ is complete and there are no negative eigenvalues in the problem; thus the linear stability of the 'parabola' solution is proved.

2.2 Other self-similar solutions

It should be emphasized that if one does not assume that $F(\xi)$ has a maximum at $\xi = 0$, the nonlinear eigenvalue problem (1.3 *a,b*) has solutions different from (1.4), (1.5). To cite an example, allow $F(\xi)$ to have local extrema inside the interval $[-1, 1]$. It follows from (1.3 *a*) that if F has a local extremum at $\xi = \xi^*$ then either $F(\xi^*) = 0$ or $F''(\xi^*) = -\dfrac{2\mu - 1}{\mu}$, which, in turns, implies either of two possibilities:

- $F(\xi)$ has only one maximum at $\xi = 0$ and then the solution of (1.3 *a,b*) is the parabola given by (1.4), (1.5); or
- $F(\xi)$ has local extrema inside the interval $[-1, 1]$ with local minima which are always equal to zero.

For instance, a solution to (1.3 *a,b*) can be found in the following form

$$F(\xi) = a\xi^{\alpha}\left(1 - \xi^{\beta}\right), \quad \alpha > 1, \quad \beta > 0, \tag{2.9}$$

where α and β are some constants. This form suggests that $F(\xi)$ has a local minimum at $\xi = 0$ with $F(0) = 0$. Substituting (2.9) into (1.3 *a*) one gets (for $1 < c < 3/2$)

$$a = \frac{c-2}{(c-1)(2c-3)}, \quad \alpha = \frac{1}{2-c}, \quad \beta = \frac{2c-3}{c-2}, \quad \mu = \frac{c-1}{2(c-2)}, \tag{2.10}$$

and for $3/2 < c < 2$

$$a = \frac{c-2}{(c-1)(3-2c)}, \quad \alpha = 2, \quad \beta = \frac{3-2c}{c-2}, \quad \mu = \frac{c-1}{2(c-2)}, \tag{2.11}$$

whence it follows that

$$F(\xi) = \frac{c-2}{(c-1)(3-2c)}\left(\xi^2 - \xi^{\frac{1}{2-c}}\right), \quad \mu = \frac{c-1}{2(c-2)} \tag{2.12}$$

for both $1 < c < 3/2$ and $3/2 < c < 2$. Notice that $F(\xi)$ given (2.12) is non-negative in the interval $[0, 1]$ since $\xi^2 - \xi^{\frac{1}{2-c}}$ is non-positive for $1 < c < 3/2$, and non-negative for $3/2 < c < 2$. Again, as μ becomes negative it is convenient to replace μ by $-\mu$ and t_0 by $-t_0$. The self-similar solution (1.2) now takes the form

$$\begin{aligned} u &= \frac{1}{2(2c-3)}B^2\,(t_0+t)^{\frac{1}{c-2}}\left[\frac{(x-x_0)^2}{B^2(t_0+t)^{\frac{c-1}{c-2}}} - \left(\frac{x-x_0}{B\,(t_0+t)^{\frac{c-1}{2(c-2)}}}\right)^{\frac{3-2c}{c-2}}\right]_+, \\ x_f &= B\,(t_0+t)^{\frac{c-1}{2(c-2)}}, \end{aligned} \tag{2.13}$$

and $u(x, t) \equiv 0$ outside the interval $[-x_f(t), x_f(t)]$. As in the case of (1.7), the time of collapse is now infinite and t_0 is an additive constant.

It should be also pointed out that for $c = 7/4$ there exists another solution of (1.3 *a,b*) in the form of (2.9), which is

$$F(\xi) = \frac{4}{3}\xi^3(1-\xi), \quad \mu = -1. \tag{2.14}$$

In this case, the self-similar solution (1.2) assumes the form

$$u = \frac{4}{3B}(x-x_0)^3\left(1 - \frac{x-x_0}{t_0+t}\right)_+, \quad x_f = B(t_0+t)^{-1}. \tag{2.15}$$

Remark: As mentioned above, the case when the solution has an isolated zero at $x = 0$ is rather complicated. The evolution in time depends on the value of the absorption coefficient c and the local behaviour of the initial data near the origin. We leave a detailed analytical and numerical study of these self-similar solutions for future research.

2.3 Numerical simulations

Numerical investigations of the evolution of non-self-similar solutions of (1.1 *a,b*) was performed in Barenblatt *et al.* [4]. A numerical procedure as well as a way to determine the constants B and t_0 were presented in this work. Assuming the quasi-steadiness of the level distribution in the vicinities of free boundaries, the speed of the propagation of the interface, which is proportional to the slope of the solution at the edge of the support, was calculated at every time step and used to evolve the numerical solution in time. The scheme was built to preserve a special property: continuity of flux at the free boundary; and, hence, for large times it yields solutions converging to the basic self-similar solutions which also satisfy this condition (see Barenblatt *et al.* [4] for details). It was shown that self-similar solution (1.6) attracts the solution of the Cauchy problem (1.1 *a,b*) having an arbitrary initial condition of compact support with a non-zero slope at the free boundary. A conjecture was made that the obtained solutions are physically relevant, but undoubtedly future research is needed to provide a definite answer to the nonuniqueness question and to understand the relation between these solutions and the 'viscosity' solution given numerically in Bertsch & Dal-Paso [5].

The analytic result (2.7) gives us an additional possibility to check the accuracy of the numerical procedure proposed in Barenblatt *et al.* [4]. Namely, we would like to calculate numerically the value of the first eigenvalue λ_0 in (2.6). To do this requires first that we determine the time t_0 when the basic self-similar solution vanishes. Given t_0, we add to the self-similar solution w_s at time $t = 0$ a small perturbation $\phi(\xi, 0)$, which is proportional to the first eigenfunction $\Phi_1(\xi) = (1 - \xi^2)^{c+1}$. We thus expect from (2.3), (2.6) that running our numerical scheme we should obtain, for instance, for small t a linear relation between $\ln(t_0 - t)$ and $\ln\left(\|w - w_s\|_\infty\right)$, i.e. in the coordinates $-\ln(t_0 - t)$, $-\ln\left(\|w - w_s\|_\infty\right)$ a straight line with a slope $\mu\lambda_0 + 4\mu - 2$. Figure (1) shows a 'log-log' plot of $\|w - w_s\|_\infty$ as a function of $t_0 - t$ up to the time when the first term in expansion (2.6) is still dominant. The solid line represents the expected result from the linear stability analysis and dots indicate data points obtained from numerical calculations. In this particular case $c = 1.75$, $\mu = 1.5$. The time t_0 has been found equal to 0.997. The parameter δ has been taken to be 0.1. One can clearly observe a linear behavior of the function. For this linear portion of the graph we have found the slope equal to 7.499, which agrees well with the analytical value 7.5 in this case. Additional experiments have been performed with different values of δ varied between 0.01 and 0.5. The slope was always found to be close to the analytical value 7.5. Naturally, with $t_0 - t$ decreasing the accuracy of the method to determine the slope is reduced.

In addition, it is instructive to investigate numerically the properties of self-similar solutions described in § 2.2. For this purpose we have taken the solution (2.13) at a certain time t as an initial condition, and computed the solution of the partial differential equation further in time. The evolution of the solution is presented in Figure 2 for different times plotted both in the coordinates x, $u(x, t)$ and in the scaled coordinates ξ, $u(\xi, t)/u_{max}(t)$. The absorption coefficient c was taken equal to 1.75. As one can see, the curves corresponding to different times collapse to the same curve it was initially, showing us that the numerical solution preserves self-similarity. In the experiment presented in Figure 2, it was very important to keep the value of $u(x, t)$ at the origin to be *exactly* zero at every time. If, due to a numerical error or any other perturbation, the value of $u(x, t)$ at $x = 0$ becomes very small ($\approx 10^{-9} - 10^{-12}$) but not zero, this solution immediately goes to the self-similar solution corresponding to 'parabola' (1.6) (see Figure 3). As it was mention above, additional theoretical and numerical work is required to clarify this point.

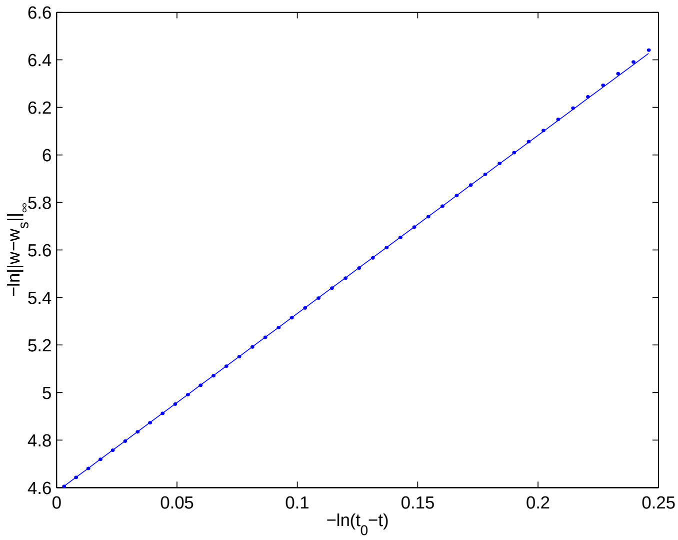

Figure 1. Determination of the numerical value of λ_0. The solid line represents the expected result from the linear stability analysis; dots indicate data points obtained from numerical calculations.

3 The axisymmetric case

3.1 A class of axisymmetric self-similar solutions

Let us consider again equation (1.1 *a*) in polar coordinates, i.e.

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \qquad (\nabla u)^2 = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2.$$

In the case of interest we have, by virtue of the axial symmetry of the problem, $u(r, \theta, t) = u(r, t)$, and equation (1.1 *a*) becomes

$$u_t = u\left[u_{rr} + \frac{1}{r}u_r\right] - (c-1)(u_r)^2. \tag{3.1}$$

By analogy with the one-dimensional case, we seek a solution of (3.1) in the self-similar form

$$u(r,t) = B^2\mu(t_0 - t)^{2\mu-1}F(\xi), \tag{3.2}$$

where the similarity variable $0 \leqslant \xi \leqslant 1$ is defined as

$$\xi = \frac{r}{B(t_0 - t)^\mu}. \tag{3.3}$$

Substituting (3.2) into (3.1) we obtain the nonlinear eigenvalue problem for the function $F(\xi)$ and for the exponent μ

$$F\frac{d^2F}{d\xi^2} + \frac{F}{\xi}\frac{dF}{d\xi} - (c-1)\left(\frac{dF}{d\xi}\right)^2 - \xi\frac{dF}{d\xi} + \frac{2\mu-1}{\mu}F = 0, \tag{3.4 a}$$

$$F'(0) = 0, \quad F(1) = 0, \quad F'(1) = -\frac{1}{c-1}. \tag{3.4 b}$$

In the special case that the function $F(\xi)$ has a maximum at $\xi = 0$, the solution of

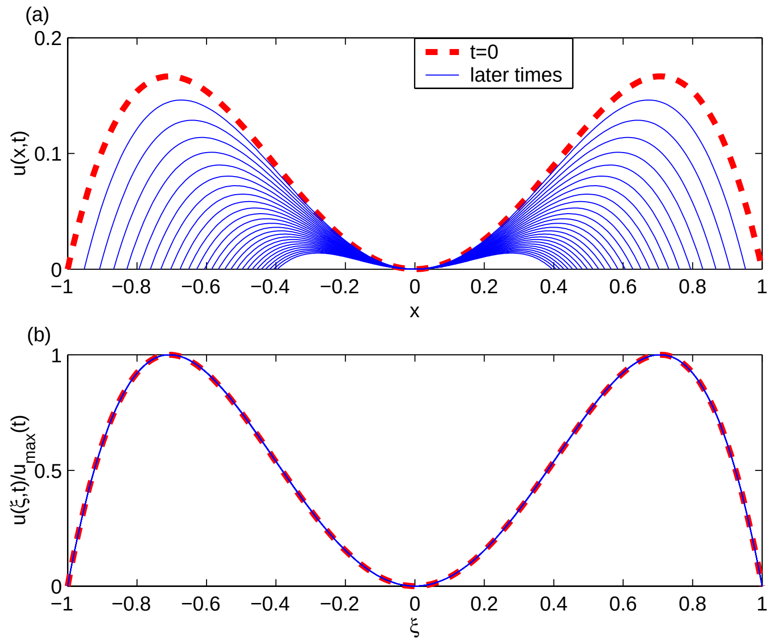

FIGURE 2. Evolution of the self-similar solution (2.13) plotted in (a) the original, non-scaled, coordinates; (b) the scaled coordinates.

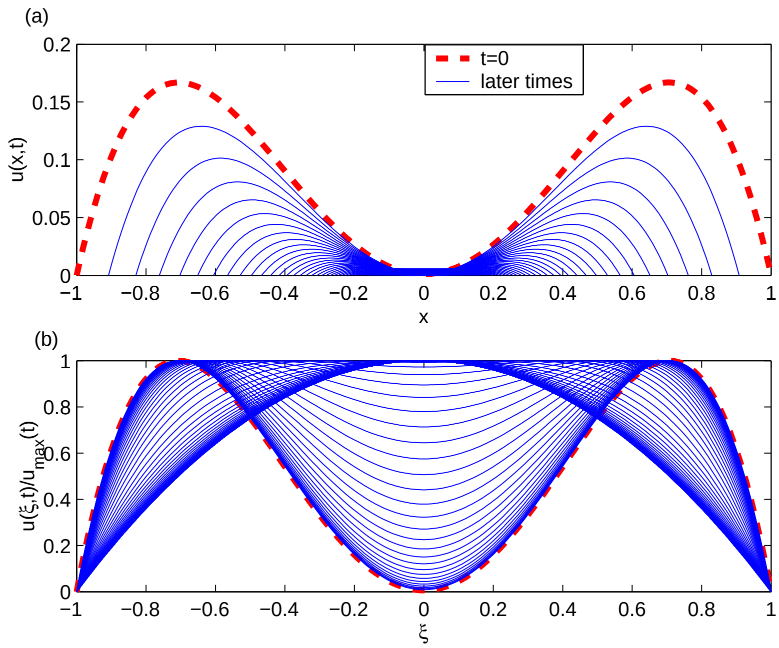

FIGURE 3. Evolution of the perturbed self-similar solution (2.13) plotted in (a) the original, non-scaled, coordinates; (b) the scaled coordinates.

(3.4a,b) can also be found analytically:

$$F(\xi) = \frac{1}{2(c-1)}\left(1-\xi^2\right), \quad \mu = \frac{c-1}{2(c-2)}. \tag{3.5}$$

The final solution for $c > 2$ so can be written in a form similar to (1.6),

$$u = \frac{B^2}{4(c-2)}(t_0 - t)^{\frac{1}{c-2}}\left[1 - \frac{r^2}{B^2(t_0-t)^{\frac{c-1}{c-2}}}\right]_+, \quad r \in [0, r_f(t)], \tag{3.6}$$

where the radius of the front is

$$r_f = B(t_0 - t)^{\frac{c-1}{2(c-2)}}, \tag{3.7}$$

and $u \equiv 0$ outside the interval $[0, r_f(t)]$. For $1 < c < 2$, μ becomes negative and we seek a solution of (3.1) in the form

$$u(r,t) = -B^2\mu(t_0+t)^{2\mu-1}F(\xi), \quad \xi = \frac{r}{B(t_0+t)^\mu}, \tag{3.8}$$

resulting in the same nonlinear eigenvalue problem (3.4a,b) and correspondingly the same $F(\xi)$ and μ. In this case, the solution assumes the form

$$u = \frac{B^2}{4(2-c)}(t_0 + t)^{\frac{1}{c-2}}\left[1 - \frac{r^2}{B^2(t_0+t)^{\frac{c-1}{c-2}}}\right]_+, \quad r_f = B(t_0+t)^{\frac{c-1}{2(c-2)}}. \tag{3.9}$$

For $c = 2$, $\mu \to \pm\infty$ and equation (3.4a) can be rewritten as

$$\frac{d}{d\xi}\left(\frac{\xi}{F}\frac{dF}{d\xi} + \frac{\xi^2}{F}\right) = 0. \tag{3.10}$$

Integration of (3.10) gives a solution satisfying boundary conditions (3.4b) in the form

$$F(\xi) = \frac{1}{\alpha-2}(\xi^2 - \xi^\alpha), \quad \alpha \neq 2,$$

where α is either equal to zero or greater than one. To write the self-similar solution (3.2) in an explicit form, by analogy with Barenblatt *et al.* [4], we represent the factors $B(t_0-t)^\mu$ and $B^2\mu(t_0-t)^{2\mu-1}$ entering the solution as

$$Bt_0^\mu\left(1-\frac{t}{t_0}\right)^\mu, \quad B^2t_0^{2\mu}\frac{\mu}{t_0}\left(1-\frac{t}{t_0}\right)^{2\mu-1},$$

respectively. Therefore, if as $c \to 2^+$ and $\mu \to \infty$, $Bt_0^\mu = C$ and $t_0 = D\mu$, we obtain for these factors

$$C\left(1-\frac{t}{D\mu}\right)^\mu \to Ce^{-t/D}, \quad \frac{C^2}{D}\left(1-\frac{t}{D\mu}\right)^{2\mu-1} \to \frac{C^2}{D}e^{-2t/D}.$$

So, solution (3.2) takes the form of the 'limiting self-similar solution'

$$u = \frac{r^2}{D(\alpha-2)}\left(1 - \frac{r^{\alpha-2}}{C^{\alpha-2}e^{-\frac{(\alpha-2)t}{D}}}\right), \tag{3.11}$$

and the radius of front decays with time according to the exponential law

$$r_f = Ce^{-t/D}. \tag{3.12}$$

For $c \to 2^-$ and $\mu \to -\infty$ self-similar solution (3.2) tends to the same limit (3.11), (3.12).

3.2 Linear stability of axisymmetric solutions

We turn now to study the linear stability of solution (3.6). As in the one-dimensional case, we investigate the stability of the smooth function $w = u^2$ which satisfies the equation

$$2\sqrt{w}w_t = 2w\Delta w - c(\nabla w)^2. \tag{3.13}$$

We consider, in accordance to the definition outlined in § 2.1, the perturbed initial-value problem, for which the initial condition at time $t = 0$ can, without loss of generality, be written in the form

$$w(r, \theta, 0) = u^2(r, 0) + \delta\phi_0(r, \theta).$$

Here δ is a small parameter, and the function $\phi_0(r, \theta)$ vanishes outside some finite interval in ξ. In the linear approximation, for $t > 0$,

$$w(r, \theta, t) = \frac{B^4}{16(c-2)^2}(t_0 - t)^{\frac{2}{c-2}}(1 - \xi^2)^2 + \delta\phi(\xi, \theta, t), \quad \xi = \frac{r}{B(t_0 - t)^{\frac{c-1}{2(c-2)}}}, \tag{3.14}$$

where $\phi(\xi, \theta, t)$ is the perturbation and $c > 2$. Due to the perturbation, the boundary is also displaced and, as in the one-dimensional case, it can be shown that the displacement is proportional to the perturbation. Substituting the perturbed solution (3.14) into (3.13) and discarding terms higher than first order in δ, we obtain for the perturbation $\phi(\xi, \theta, t)$ the equation

$$\begin{aligned}\phi_\tau = L_\xi = (1-\xi^2)\phi_{\xi\xi} + \frac{1-\xi^2}{\xi^2}\phi_{\theta\theta} + \frac{1-\xi^2}{\xi}\phi_\xi + 2(c+1)\xi\phi_\xi \\ + 4(c+1)\frac{\phi}{1-\xi^2} - 4(c+2)\phi,\end{aligned} \tag{3.15}$$

where $\tau = -\frac{1}{4(c-2)}\ln(t_0 - t)$. Again, both the function ϕ and its first derivative with respect to ξ must be continuous at the edge of the support. To determine a solution of the initial-value problem for the perturbation, we apply the method of separation of variables to (3.15). Looking for product solutions yields

$$\begin{aligned}\phi(r, \theta, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(t_0 - t)^{\frac{\lambda_{mn}}{4(c-2)}}\cos m\theta\ \Phi_{mn}(\xi) \\ + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}(t_0 - t)^{\frac{\lambda_{mn}}{4(c-2)}}\sin m\theta\ \Phi_{mn}(\xi),\end{aligned} \tag{3.16}$$

where $\Phi_{mn}(\xi)$ is the eigenfunction corresponding to the eigenvalue λ_{mn}. To prove linear stability of the self-similar solution (3.6) we should show that all eigenvalues of (3.15) are positive and the system of the eigenfunction $\{\Phi_{mn}(\xi)\}$ is complete. As in the one-dimensional case, the eigenfunctions of the operator (3.15) can be expressed in terms of Jacobi polynomials. Here we have

$$\lambda_{mn} = 4n(m + c + 1 + n) + 2m(c + 1) + 4c + 8, \tag{3.17}$$

and

$$\Phi_{mn}(\xi) = \xi^m(1 - \xi^2)^{c+1}P_n^{(m+c+1,c+1)}(\xi^2). \tag{3.18}$$

The Jacobi polynomials $P_n^{(m+c+1,c+1)}(x)$ represent a complete system and are defined in general by

$$P_n^{(m+c+1,c+1)}(x) = \frac{x^{-m}(1-x)^{-c}}{(m+1)(m+2)\cdots(m+n)} \frac{d^n}{dx^n}\left[x^{m+n}(1-x)^{c+n}\right],$$

or can be calculated much like in the one-dimensional case by using a reccurence formula (see Szegö [14]).

It follows from (3.17) that all eigenvalues of problem (3.15) are positive, which proves the linear stability of the self-similar solution for $c > 2$. A similar result can be easily obtained for the case $1 < c < 2$ by replacing μ by $-\mu$ and t_0 by $-t_0$.

3.3 Numerical simulations

The goal of the numerical simulations was to demonstrate that the self-similar solution obtained above attracts the solution of non-self-similar Cauchy problems having an arbitrary axisymmetric initial condition of compact support. To simplify numerical calculations we make a change of variables going from the time dependent interval $r \in [0, r_f(t)]$ to the fixed interval $\xi \in [0, 1]$, namely,

$$\xi = \frac{r}{r_f(t)},$$

where $r_f(t)$ is the radius of the front at time t. With this substitution equation (3.1) becomes

$$u_t = \frac{\dot{r}_f(t)\xi}{r_f(t)}\, u_\xi + \frac{1}{r_f^2(t)}\left[u\left(u_{\xi\xi} + \frac{1}{\xi}u_\xi\right) - (c-1)(u_\xi)^2\right], \quad 0 \leqslant \xi \leqslant 1, \tag{3.19}$$

indicating that we need an additional equation for $\dot{r}_f(t) = dr_f(t)/dt$. Keeping in mind that the solution is quasi-stationary at $r_f(t)$ yields

$$\dot{r}_f(t) = (c-1)u_r(r_f(t), t) = (c-1)\frac{u_\xi(1,t)}{r_f(t)}, \tag{3.20}$$

and we finally obtain

$$u_t = \frac{1}{r_f^2(t)}\left[u\left(u_{\xi\xi} + \frac{1}{\xi}u_\xi\right) - (c-1)(u_\xi)^2 + (c-1)\xi u_\xi(1,t)u_\xi\right], \tag{3.21 a}$$

with the initial condition

$$u(\xi, 0) = \begin{cases} u_0(\xi), & 0 \leqslant \xi \leqslant 1, \\ 0\ , & \xi \geqslant 1, \end{cases} \tag{3.21 b}$$

and the boundary conditions

$$u_\xi(0,t) = 0\ , \qquad u(1,t) = 0, \qquad t \geqslant 0. \tag{3.21 c}$$

The finite-difference scheme used for solving (3.21 *a*-*c*) is similar to one used in Barenblatt *et al.* [4] for the one-dimensional case and is given in the Appendix. In all simulations the number N of subintervals was equal to 200. The time step Δt was chosen small enough to ensure local stability of the method. The absorption coefficient c was always equal to 2.25.

The first computation was performed for an ‘arbitrary’ axisymmetric initial condition.

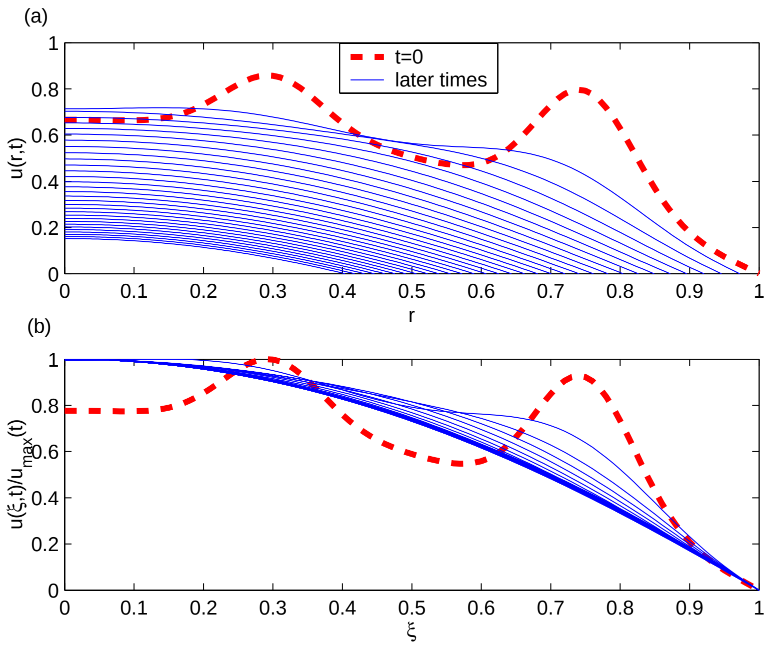

FIGURE 4. Evolution of the arbitrary axisymmetric initial distribution plotted in (a) the original, non-scaled, coordinates; (b) the scaled coordinates.

Figure 4 shows the plot of the numerical solution both in the coordinates r, $u(r,t)$ and in the scaled coordinates ξ, $u(\xi,t)/u_{max}(t)$ for different times. As time increases the numerical solution tends to the self-similar asymptotics corresponding to (3.6). From numerical experiments it was found that for this initial condition $t_0 = 1.582$, $B = 0.305$, and the numerical value of the exponent $\mu = 2.500$ which agrees well with the analytical value.

For comparison we took the self-similar solution (3.6) as an initial condition, and computed the numerical solution of the partial-differential equation further in time. The results are presented in Figure 5 for different times (again, both in the coordinates r, $u(r,t)$ and in the scaled coordinates ξ, $u(\xi,t)/u_{max}(t)$). Being plotted in scaled coordinates, all curves collapse to a single curve, lending additional support for the validity on the numerical algorithm.

4 Conclusions

We considered the one- and two-dimensional filtration-absorption equations. A family of axisymmetric self-similar solutions has been constructed by the analogy with the one-dimensional case studied in Barenblatt *et al.* [4]. Numerical experiments indicate that the self-similar solutions obtained attract the solutions of non-self-similar Cauchy problems having the initial condition of compact support. It has been proved analytically both in the one-dimensional case and in the two-dimensional case that the basic self-similar solutions are stable with respect to small perturbations. It has been also shown that self-similar solutions of the filtration-absorption equation are not uniquely determined. Self-similar

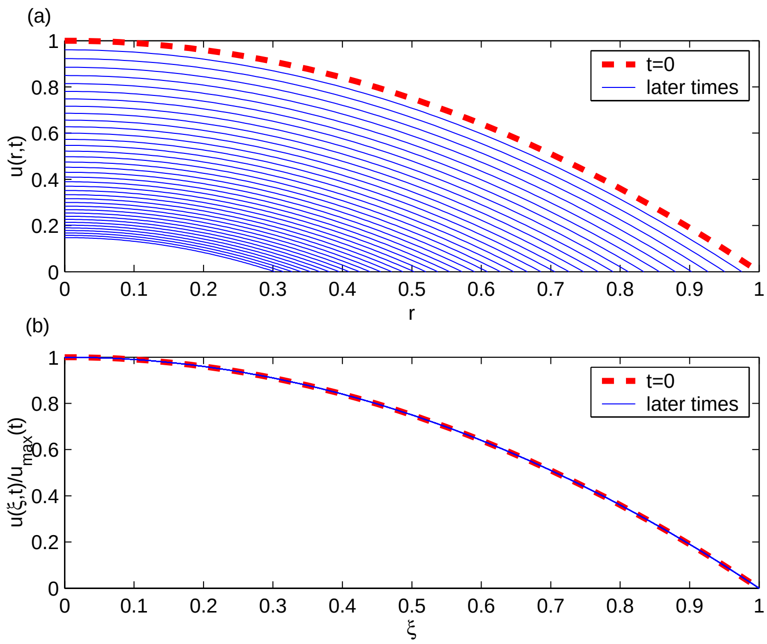

FIGURE 5. Evolution of self-similar solution (3.6) plotted in (a) the original, non-scaled, coordinates; (b) the scaled coordinates.

solutions which are different from solutions obtained in Barenblatt *et al.* [4], but satisfy the same nonlinear eigenvalue problem have been found in the one-dimensional case. Based on numerical experiments it has been suggested that the additional self-similar solutions are structurally unstable. Additional theoretical and numerical work is needed, however, to clarify the last point and to provide a definite answer to the nonuniqueness question.

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Appendix A

To solve (3.21 *a-c*) numerically we introduce the uniform grid

$$\xi_i = i\Delta\xi, \quad i = 0 \dots N, \quad \Delta\xi N = 1.$$

Using the notation

$$t_n = n\Delta t, \qquad u_i^n = u_i(t_n) = u(\xi_i, t_n), \qquad x_f^n = x_f(t_n),$$

we replace (3.21 *a*) by the second order approximation

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{(r_f^n)^2}\left[u_i^n\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta\xi^2} + \frac{1}{\xi_i}u_i^n u_{\xi,i}^n - (c-1)(u_{\xi,i}^n)^2 + (c-1)\xi_i u_{\xi,N}^n u_{\xi,i}^n\right], \quad i=\overline{1,N-1}. \tag{A 1}$$

Here

$$u_{\xi,i}^n = \frac{u_{i+1}^n-u_{i-1}^n}{2\Delta\xi}, \quad u_{\xi,N}^n = \frac{-4u_{N-1}^n+u_{N-2}^n}{2\Delta\xi}. \tag{A 2}$$

To obtain a second order approximation for the boundary condition at $\xi=0$ we use the expansion

$$u_{\xi,0}^n = \frac{u_1^n-u_0^n}{\Delta\xi} = \frac{\partial u(0,t_n)}{\partial\xi} + \frac{\Delta\xi}{2}\frac{\partial^2 u(0,t_n)}{\partial\xi^2} + O(\Delta\xi^2). \tag{A 3}$$

Next, we want to exclude $\frac{\partial^2 u(0,t_n)}{\partial\xi^2}$ from (A 3). Taking the limit $\xi\to 0$ in (3.21 *a*) and using the fact that $\lim_{\xi\to 0}\frac{u_\xi(\xi,t)}{\xi} = u_{\xi\xi}(0,t)$ we obtain

$$\frac{\partial u(0,t)}{\partial t} = \frac{2u(0,t)}{r_f^2(t)}\frac{\partial^2 u(0,t)}{\partial\xi^2}.$$

It immediately follows that

$$\frac{\partial u(0,t_n)}{\partial\xi} = u_{\xi,0}^n - \frac{\Delta\xi}{4}\frac{r_f^2(t)}{u(0,t)}\frac{\partial u(0,t)}{\partial t} + O(\Delta\xi^2).$$

Hence at $\xi=0$ we have

$$\frac{u_0^{n+1}-u_0^n}{\Delta t} = \frac{4u_0^n}{(r_f^n)^2}\frac{u_{\xi,0}}{\Delta\xi}. \tag{A 4}$$

Finally, the approximation of (3.20), (3.21 *a-c*) assumes the form

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{(r_f^n)^2}\left[u_i^n\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta\xi^2} + \frac{1}{\xi_i}u_i^n u_{\xi,i}^n - (c-1)(u_{\xi,i}^n)^2 + (c-1)\xi_i u_{\xi,N}^n u_{\xi,i}^n\right], \quad i=\overline{1,N-1}, \tag{A 5a}$$

$$\frac{u_0^{n+1}-u_0^n}{\Delta t} = \frac{4u_0^n}{(r_f^n)^2}\frac{u_{\xi,0}}{\Delta\xi}, \quad u_N^{n+1}=0, \tag{A 5b}$$

$$\frac{r_f^{n+1}-r_f^n}{\Delta t} = \frac{c-1}{r_f^n}u_{\xi,N}^n, \tag{A 5c}$$

where $u_{\xi,0}^n = \frac{u_1^n-u_0^n}{\Delta\xi}$ and $u_{\xi,i}^n$ and $u_{\xi,N}^n$ are given by (A 2).

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