

# Particle Methods for Dispersive Equations

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We introduce a new *dispersion-velocity* particle method for approximating solutions of linear and nonlinear dispersive equations. This is the first time in which particle methods are being used for solving such equations. Our method is based on an extension of the diffusion-velocity method of Degond and Mustieles (*SIAM J. Sci. Stat. Comput.* **11**(2), 293 (1990)) to the dispersive framework. The main analytical result we provide is the short time existence and uniqueness of a solution to the resulting dispersion-velocity transport equation. We numerically test our new method for a variety of linear and nonlinear problems. In particular we are interested in nonlinear equations which generate structures that have nonsmooth fronts. Our simulations show that this particle method is capable of capturing the *nonlinear* regime of a compacton–compacton type interaction. © 2001 Academic Press

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## 1. INTRODUCTION

In recent years, particle methods have become one of the most useful and widespread tools for approximating solutions of partial differential equations in a variety of fields. In these methods, a solution of a given equation is represented by a collection of particles, located in points  $x_i$  and carrying masses  $w_i$ . Equations of evolution in time are then written to describe the dynamics of the location of the particles and their weights. Due to the Lagrangian nature of the method, small scales that might develop in a solution can be easily described with a relatively small number of particles. This property is what made particle methods so attractive in practice.

In this work we present the first particle method for approximating solutions of linear and nonlinear dispersive equations. Our method is based on the diffusion-velocity method, which was introduced in [11] for approximating solutions of parabolic equations, and we therefore name our new method the *dispersion-velocity method*. The dispersion-velocity

method is the first particle method to be proposed per se for approximating solutions of such equations. Most importantly, this is the first attempt to use particles for directly simulating interactions between solitary waves.

Since our starting point was a particle method for parabolic equations, we briefly describe some of the ideas that are used for such equations. It is generally possible to divide the particle methods for approximating parabolic equations into two classes: stochastic methods and deterministic methods.

The most widely used treatment of diffusion terms, the *random vortex method*, was introduced by Chorin in [6]. There, diffusion was introduced by adding a Wiener process to the motion of each vortex. Numerous works followed that pioneering paper (see, e.g., [1–4, 15, 18–20, 29, 31]). For a comprehensive list we refer to the review paper of Puckett [32] and the book by Cottet and Koumoutsakos [8].

A different approach in which particle methods were used for approximating solutions to the heat equation and related models (such as the Fokker–Planck equation, a Boltzmann-like equation—the Kac equation and Navier–Stokes (NS) equations), was introduced by Russo in [38, 39]. In these works, the diffusion of the particles was described as a deterministic process in terms of a mean motion with a speed equal to the osmotic velocity associated with the diffusion process. In a following work [40], the method was shown to be successful for approximating solutions to the two-dimensional Navier–Stokes (NS) equation in an unbounded domain. In this setup, the particles were convected according to the velocity field while their weights evolved according to the diffusion term in the vorticity formulation of the NS equations. See also Fishelov [13] and Mas–Gallic and Raviart [30].

Another deterministic approach for approximating solutions of the parabolic equations with particle methods was introduced by Degond and Mustieles in [11]. Their so-called *diffusion-velocity* method was based on defining the convective field associated with the heat operator which then allowed the particles to convect in a standard way.

For example, the one-dimensional heat equation

$$u_t = u_{xx}$$

is rewritten as

$$u_t + (a(u)u)_x = 0,$$

where the velocity  $a(u)$  is taken as  $-u_x/u$ . Particles carrying fixed masses will be then convected with speed  $a(u)$ . The convergence properties of the diffusion-velocity method were investigated, e.g., in [24, 25], where short time existence and uniqueness of solutions to the resulting diffusion-velocity transport equation were proved. The diffusion-velocity method serves as the basic tool for the derivation of our particle methods in the dispersive world.

We focus our attention on linear and nonlinear dispersive partial differential equations. Our model problem in the linear setup is the linear Airy equation,

$$u_t = u_{xxx}.$$

The success of particle methods in approximating the oscillatory solutions that develop in this dispersive equation provides us with valuable insight regarding the potential embedded in our approach.

In the nonlinear setup, we focus on equations which generate compactly supported solutions with nonsmooth fronts, the prototype being the  $K(m, n)$  equation, which was introduced by Rosenau and Hyman in [34]. In this equation, a nonlinear dispersion term replaces the linear dispersion term in the Korteweg-de Vries (KdV) equation, resulting with

$$K(m, n): u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, 1 < n \leq 3.$$

For certain values of  $m$  and  $n$ , the  $K(m, n)$  equation has solitary waves which are compactly supported. In particular, the variant  $K(2, 2)$ ,

$$K(2, 2): u_t + (u^2)_x + (u^2)_{xxx} = 0,$$

has a fundamental “compacton” solution of the form

$$u(x, t) = \frac{4\lambda}{3} \left[ \cos\left(\frac{x - \lambda t}{4}\right) \right]^2, \quad |x - \lambda t| \leq 2\pi.$$

After the first appearance of the compactons in [34], it turned out that similar structures emerge as solutions for a much larger class of nonlinear PDEs (see [26, 27, 35, 36]), among which is, e.g.,

$$u_t + (u^m)_x + (u(u^n)_{xx})_x = 0, \quad m > 1, \quad m = n + 1,$$

which we consider with  $m = 2, n = 1$  as our nonlinear model problem.

In this work we are mainly interested in developing tools for approximating numerically solutions to equations which generate nonsmooth structures. Due to the discontinuity in the derivatives on the fronts of these emerging structures, standard numerical methods such as finite-differences and pseudo-spectral methods generate spurious oscillations on the fronts. Controlling these oscillations calls for a numerical filtering of the higher modes, which might result in the elimination of fine scales from the solution. Moreover, in cases where a positive solution should remain positive in time, the spurious numerical oscillations might cause the solution to change sign. In this case, one can fall into an ill-posed region of the equation, and the numerical solution will cease to represent the solution to the equation at hand (see the discussion in [14]).

Finally, we would like to comment that there have been several attempts to address the difficulties in approximating solutions of compacton equations. In [14] and [22], e.g., solutions to the  $K(2, 2)$  equation were obtained with finite-difference methods. These methods were shown to generate instabilities on the discontinuous fronts (which were interpreted in [14] as shocks). In [34], the solution of compacton equations was generated by pseudo-spectral approximations while filtering out the high modes. None of these works presented a study of the properties of the numerical scheme used.

The structure of the paper is as follows: we start in Section 2 by introducing the new dispersion-velocity method in the context of linear equations. The main analytical result in this section is Theorem 2.1, where we prove (in the spirit of [25]) a short time existence and uniqueness for solutions of the dispersion-velocity transport equation. This theorem requires the initial data to have only one bounded derivative and provides the same regularity for the resulting solution.

In Section 3 we show how to make the adjustments required in order to adapt our dispersion-velocity methods to nonlinear problems. Following the discussion above, the derivation of our method is done on compacton-type equations, which develop structures with nonsmooth interfaces.

Our numerical method is summarized in Section 4. For completeness we discuss several issues relating to various aspects of the implementation of the method, such as, e.g., the initialization, the cutoff functions, and the accuracy of the method.

We conclude in Section 5 with several numerical examples for linear and nonlinear equations. In the linear examples we are able to verify the accuracy and the  $L^2$  conservation properties of the scheme. In the nonlinear examples we show that the particles that are spread over two compactons (moving with different velocities) are capable of going through the nonlinear compacton–compacton interaction and emerge from the interaction, while preserving the phase shift which is typical with this type of interaction.

## 2. THE DISPERSION-VELOCITY METHOD: LINEAR PROBLEMS

In this section we present the new *dispersion-velocity* method for approximating solutions of linear dispersive equations. Extension of this method to nonlinear problems will be presented in Section 3 below.

The *dispersion-velocity* method is based on the *diffusion-velocity* method which was introduced by Degond and Mustieles in [11]. There, a deterministic particle method was used to approximate solutions to the linear heat equation,  $u_t - \nabla \cdot (S(x, t) \cdot \nabla u) = 0$ , by rewriting it as an advection equation,  $u_t + \nabla \cdot (A(x, t)u) = 0$ , and advecting particles with a speed  $A(x, t) = -S(x, t) \cdot \nabla u(x, t)/u(x, t)$ .

Our starting point is the scalar, linear dispersive equation in one space dimension,

$$u_t = u_{xxx}, \quad (2.1)$$

subject to the initial data  $u(x, t = 0) = u_0(x)$ . Boundary conditions will be specified below.

One can rewrite Eq. (2.1) as a convection equation

$$u_t + (a(x, t)u)_x = 0, \quad (2.2)$$

where the coefficient  $a(x, t)$  in (2.2) has to satisfy

$$a(x, t)u(x, t) = -u_{xx}(x, t),$$

which, in turn, leads to

$$a(x, t) = -\frac{u_{xx}(x, t)}{u(x, t)}. \quad (2.3)$$

If  $a(x, t)$  is a known function, then (2.2) is a convection equation. A “standard” particle method for approximating solutions to (2.3) when  $a(x, t)$  is known is based on introducing a distribution of the form

$$u_N(x, t) = \sum_{i=1}^N w_i \delta(x - x_i(t)),$$

where the initial data is approximated by

$$u_N(x, 0) = \sum_{i=1}^N w_i \delta(x - x_i(0)) \simeq u_0(x).$$

Here  $x_i(t)$  is the characteristic curve associated with  $a(x, t)$ , which starts at the point  $x_i^0$ ; i.e.,

$$\begin{cases} \frac{dx_i}{dt} = a(x_i(t), t), \\ x_i(0) = x_i^0. \end{cases} \tag{2.4}$$

According to (2.3),  $a(x, t)$  depends on  $u$  and on its second derivative,  $u_{xx}$ , and, therefore, it cannot be considered as a given function. Moreover, since the product of  $\delta$  functions is not well defined, the standard particle method has to be modified.

Following [11], we introduce a smoothed approximation,  $u_N^\epsilon(x, t)$ ,

$$u_N^\epsilon(x, t) = (u_N * \zeta_\epsilon)(x, t) = \sum_{i=1}^N w_i \zeta_\epsilon(x - x_i(t)). \tag{2.5}$$

The function  $\zeta_\epsilon(x)$  (which is also called ‘‘cutoff function’’) is taken as a smooth approximation of the  $\delta$  function which satisfies

$$\zeta_\epsilon(x) = \frac{1}{\epsilon} \zeta\left(\frac{x}{\epsilon}\right), \quad \text{and} \quad \int \zeta(x) dx = 1. \tag{2.6}$$

Given an appropriate smoothing function  $\zeta_\epsilon(x)$ , we can approximate  $a(x, t)$  in (2.3) by

$$a_\zeta(x, t) = -\frac{u * \zeta_\epsilon''}{u * \zeta_\epsilon}, \tag{2.7}$$

resulting with the *dispersion-velocity transport equation*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a_\zeta u) = 0, \\ u(x, t = 0) = u_0(x). \end{cases} \tag{2.8}$$

The resulting *dispersion-velocity method* is obtained by considering a particle approximation as a distribution of the form (2.5), where  $x_i(t)$  are the solutions to

$$\begin{cases} \frac{dx_i}{dt} = -\frac{(u_N^\epsilon(x_i, t))''}{u_N^\epsilon(x_i, t)} = -\frac{\sum_{j=1}^N w_j \zeta_\epsilon''(x_i - x_j)}{\sum_{j=1}^N w_j \zeta_\epsilon(x_i - x_j)}, \\ x_i(0) = x_i^0. \end{cases} \tag{2.9}$$

Local existence and uniqueness of a solution to the system of ODEs, (2.9), result from standard ODE theorems. In order to switch from the solution along these characteristics back to the solution to the dispersion-velocity transport equation (2.8), one typically requires certain regularity of the equation and the initial data. More specifically, if a first-order (nonlinear) PDE is written as  $F(t, x, u, u_x, u_t) = 0$ , a standard requirement is that  $F$  will have a continuous second-order derivative with respect to its arguments (see [12, 23]).

In our case, such a condition will amount to requiring, e.g., that the initial data,  $u_0$ , has three continuous derivatives. While this might be acceptable in the linear case, it will be unacceptable in the nonlinear case, where we will be interested in initial data that has only one derivative.

The following theorem provides a short time existence and uniqueness of a solution to the dispersive-velocity transport equation (2.8) under the assumption that the initial data has only one bounded derivative. The proof follows the arguments of Lacombe and Mas-Gallic [25] for the diffusion-velocity transport equation (see also [24]). Here, however, we improve the result of [25] by observing that the resulting solution has the same regularity as the initial data.

**THEOREM 2.1 (Local Existence and Uniqueness).** *Assume  $\zeta_\epsilon \in C^4(\mathbf{R})$ ,  $u_0 \in W^{1,\infty}(\mathbf{R})$ , and that there exist constants  $\alpha, \beta > 0$  such that  $\alpha \leq u_0 \leq \beta$ . Then there exists  $T_0$  such that (2.8) has a unique solution in  $W^{1,\infty}(\mathbf{R} \times (0, T_0))$ .*

*Proof.* The proof follows the arguments of [25] with the required adaptations to the dispersive framework and additional bootstrapping arguments regarding the regularity of the solution. It is based on a fixed point argument on the functional  $\phi \in L^\infty(\mathbf{R} \times (0, T))$  that maps any  $V \in L^\infty(\mathbf{R} \times (0, T))$  to the unique solution to the linear advection equation,  $v$ , by

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(va_\zeta(V)) = 0, \\ v(x, t = 0) = u_0(x); \end{cases} \quad (2.10)$$

namely, for every suitable  $V$ , the unique solution of (2.10) is denoted by  $v = \phi(V)$ . Due to the smoothness of  $a_\zeta(V)$ , given  $u_0 \in W^{1,\infty}$ ,  $v$  is also in  $W^{1,\infty}$ . Utilizing the method of characteristics, the solution of (2.10) can be written as

$$v(x, t) = \phi(V)(x, t) = u_0(X(0)) \exp\left(-\int_0^t a'_\zeta(V)(X(s), s) ds\right), \quad (2.11)$$

where the characteristic curve  $X(s)$  is the solution of

$$\begin{cases} \frac{dX}{dt} = a_\zeta(V)(X, t), \\ X(t = 0) = x. \end{cases} \quad (2.12)$$

We now let  $\mathcal{A}$  denote the set of functions in  $L^\infty$  which are bounded in a strip away from the origin,

$$\mathcal{A} = \{u \in L^\infty(\mathbf{R} \times (0, T)) : \alpha e^{-1} \leq u \leq \beta e\}, \quad \alpha, \beta > 0.$$

In order to complete our proof, all that is required is to prove that  $\mathcal{A}$  is stable under  $\phi$ , i.e.,  $\phi(\mathcal{A}) \subseteq \mathcal{A}$ , and that  $\phi$  is a strict  $L^\infty$  contraction on  $\mathcal{A}$  (both results will be shown to hold for a short time).

First, given  $V \in \mathcal{A}$  we would like to show that  $\phi(V) \in \mathcal{A}$ . We denote the  $L^1$  norms of  $\zeta_\epsilon$  and its derivatives by  $c_i = \|\zeta_\epsilon\|_{W^{i,1}}$ ,  $i = 0, 1, \dots$ , where due to the normalization ( $\int \zeta_\epsilon = 1$ ), the first constant,  $c_0$ , equals 1. With this notation, the derivative of  $a_\zeta$  can be estimated by

$$|a'_\zeta(V)| = \left\| \frac{(V * \zeta'_\epsilon)(V * \zeta''_\epsilon) - (V * \zeta''_\epsilon)(V * \zeta_\epsilon)}{(V * \zeta_\epsilon)^2} \right\|_\infty \leq \frac{\beta^2 e^4 (c_3 + c_1 c_2)}{\alpha^2} := \frac{1}{T_1}. \quad (2.13)$$

Hence, for  $T \leq T_1$ ,

$$e^{-1} \leq e^{-\int_0^T (a_\zeta)'} \leq e,$$

and therefore by (2.11) one can conclude that since  $\alpha \leq u_0 \leq \beta$ ,  $\phi(V) \in \mathcal{A}$  which ends the first part of the proof.

In order to proceed, we take  $U, V \in \mathcal{A}$ , such that  $u = \phi(U)$  and  $v = \phi(V)$ . We will prove that  $\phi$  is a contraction in  $L^\infty$ , namely, that there exists a constant  $L < 1$  and a time  $\tilde{T}$  such that  $\forall T < \tilde{T}$

$$\|\phi(U) - \phi(V)\|_\infty \leq L\|U - V\|_\infty.$$

Clearly, the difference  $w = u - v$  satisfies

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(w a_\zeta(V)) = f, \\ w(x, t = 0) = 0, \end{cases} \tag{2.14}$$

where

$$f = (u[a_\zeta(U) - a_\zeta(V)])'.$$

Once again, using the method of characteristics, the solution to (2.14) can be written as

$$w(x, t) = \int_0^t J(\tau, x, t) f(\tau, x) d\tau,$$

where

$$J(t, x, s) = \exp\left(-\int_s^t a'_\zeta(V)(X(\sigma), \sigma) d\sigma\right),$$

and the characteristic curve  $X$  is given by (2.12).

Since  $V \in \mathcal{A}$ , it follows from (2.13) that  $|(a_\zeta(V))'| \leq 1/T_1$ , and hence for  $T \leq T_1$ ,  $|J| \leq e$ , which, in turn, carries  $\|w\|_\infty \leq T e \|f\|_\infty$ . All that is left is to bound  $f$ , an estimate which will be obtained in two steps. We start by bounding

$$\|f\|_\infty \leq \|u'[a_\zeta(U) - a_\zeta(V)]\|_\infty + \|u[a'_\zeta(U) - a'_\zeta(V)]\|_\infty := \mathcal{I}_1 + \mathcal{I}_2. \tag{2.15}$$

Since the difference  $a_\zeta(U) - a_\zeta(V)$  can be rewritten as

$$a_\zeta(U) - a_\zeta(V) = \frac{(V * \zeta'_\epsilon)[(U - V) * \zeta_\epsilon] - [(U - V) * \zeta''] (V * \zeta_\epsilon)}{(U * \zeta_\epsilon)(V * \zeta_\epsilon)},$$

the first term on the RHS of (2.15),  $\mathcal{I}_1$ , is bounded by

$$\mathcal{I}_1 \leq \|u'\|_\infty \frac{2\beta e^3 c_2}{\alpha^2} |U - V|,$$

which still leaves us with the task of bounding  $\|u'\|_\infty$ :

$$\|u_x\|_\infty \leq \left\| u'_0 e^{-\int a'_\zeta(U)} \right\|_\infty + \left\| -u_0 \int a''_\zeta(U) e^{-\int a'_\zeta(U)} \right\|_\infty := \mathcal{I}_{11} + \mathcal{I}_{12}. \tag{2.16}$$

The first term on the RHS of (2.16),  $\mathcal{I}_{11}$ , can be bounded by

$$\mathcal{I}_{11} \leq \|u'_0\|_\infty \left\| e^{-\int a'_\zeta(U)} \right\| \leq e \|u_0\|_{W^{1,\infty}}.$$

We also have

$$\begin{aligned} \|a''_\zeta(U)\|_\infty &\leq \left\| \frac{U * \zeta_\epsilon^{(4)}}{U * \zeta_\epsilon} \right\|_\infty + \left\| \left( \frac{U * \zeta''_\epsilon}{U * \zeta_\epsilon} \right)^2 \right\|_\infty + 2 \left\| \frac{(U * \zeta'''_\epsilon)(U * \zeta'_\epsilon)}{(U * \zeta_\epsilon)^2} \right\|_\infty \\ &\quad + 2 \left\| \frac{(U * \zeta''_\epsilon)(U * \zeta'_\epsilon)^2}{(U * \zeta_\epsilon)^3} \right\|_\infty, \end{aligned}$$

and therefore for the second term of the RHS of (2.16),  $\mathcal{I}_{12}$ , we have

$$\mathcal{I}_{12} \leq \beta e T \|a''_\zeta(U)\|_\infty \leq \beta e T \left[ \frac{\beta e^2 c_4}{\alpha} + \frac{\beta^2 (c_2^2 + 2c_1 c_3) e^4}{\alpha^2} + 2 \frac{\beta^3 e^6 c_1^2 c_2}{\alpha^3} \right],$$

from which we can conclude that for  $T \leq 1$ ,

$$\mathcal{I}_1 \leq K_1 |U - V|, \quad (2.17)$$

where

$$K_1 = \frac{2\beta e^3 c_2}{\alpha^2} \left[ e \|u_0\|_{W^{1,\infty}} + \frac{\beta^2 e^3}{\alpha} \left( c_4 + \frac{\beta e^2 (c_2^2 + 2c_1 c_3)}{\alpha} + 2 \frac{\beta^2 e^4 c_1^2 c_2}{\alpha^2} \right) \right].$$

We are now ready to estimate the second term on the RHS of (2.15),  $\mathcal{I}_2$ . First, we rewrite the difference  $a'_\zeta(U) - a'_\zeta(V)$  as

$$\begin{aligned} a'_\zeta(U) - a'_\zeta(V) &= \frac{(V * \zeta'''_\epsilon)[(U - V) * \zeta_\epsilon] - [(U - V) * \zeta'''_\epsilon](V * \zeta_\epsilon)}{(U * \zeta_\epsilon)(V * \zeta_\epsilon)} \\ &\quad + \frac{(U * \zeta'_\epsilon)[(U - V) * \zeta''_\epsilon]}{(U * \zeta_\epsilon)^2} + \frac{[(U - V) * \zeta'_\epsilon](V * \zeta''_\epsilon)}{(V * \zeta_\epsilon)^2} \\ &\quad - \frac{(V * \zeta''_\epsilon)(U * \zeta'_\epsilon)[(U + V) * \zeta_\epsilon] + [(U - V) * \zeta_\epsilon]}{(U * \zeta_\epsilon)^2 (V * \zeta_\epsilon)^2}. \end{aligned}$$

Hence

$$\mathcal{I}_2 \leq \|u\|_\infty \|a'_\zeta(U) - a'_\zeta(V)\|_\infty \leq K_2 |U - V|, \quad (2.18)$$

with

$$K_2 = \frac{2\beta^2 e^4}{\alpha^2} \left( c_3 + c_1 c_2 + c_1 c_2 \frac{\beta^2 e^4}{\alpha^2} \right).$$

Combining the estimates (2.17) and (2.18) we can finally conclude that

$$\|\phi(U) - \phi(V)\|_\infty = \|w\|_\infty \leq T e \|f\|_\infty \leq T e (\mathcal{I}_1 + \mathcal{I}_2) \leq T K |U - V|,$$

where  $K = K_1 + K_2$ . The mapping  $\phi$  is therefore a contraction in  $L^\infty$  assuming that  $T < \min(T_1, 1/K, 1)$ , which guarantees that it has a unique fixed point,  $\tilde{V} = \phi(\tilde{V}) \in \mathcal{A}$ . Since  $\phi$  maps every element of  $\mathcal{A}$  to a solution of the PDE (2.10), it also maps the fixed point of  $\phi$ ,  $\tilde{V}$  to a solution of (2.10), and hence  $\tilde{V} = \phi(\tilde{V}) \in W^{1,\infty}$ . This concludes the proof.

*Remarks.* 1. It is straightforward to extend the results of Theorem 2.1 to equations of the type

$$u_t + (bu)_x = u_{xxx}.$$

2. The results of Theorem 2.1 also hold for periodic boundary conditions, with the suitable adjustments in the values of the constants,  $c_i$ ,  $i = 1, 2, \dots$ .

3. We would like to emphasize that Theorem 2.1 does not imply the stability or the convergence of the numerical scheme (2.9). The existence time provided by the theorem tends to zero as  $\epsilon$  tends to zero, which is the limit in which one would like the scheme to converge (together with  $N \rightarrow \infty$ ). Convergence would therefore require a strong result of existence and boundedness for a period of time that does not go to zero with  $\epsilon$ .

### 3. THE DISPERSION-VELOCITY METHOD: NONLINEAR PROBLEMS

In this section we show how the *dispersion-velocity* method can be used for approximating solutions of equations with nonlinear dispersion terms. We would like to demonstrate the advantages of our new techniques when compared with traditional finite-differences methods which lead us to start our research by focusing on problems which develop non-smooth fronts and are therefore difficult to solve numerically. We would like to stress that our methods are not limited to such equations only. They can be applied to a variety of other interesting problems, some of which we will comment on in the remarks below. To this extent, we consider the nonlinear dispersive equation,

$$u_t + (u^2)_x + (uu_{xx})_x = 0, \quad (3.1)$$

subject to initial data  $u(x, t = 0) = u_0(x)$ . In this case, the ‘‘compacton’’ which is the fundamental solution to (3.1) has the compact form (see [35]),

$$u(x, t) = 2\lambda \left[ \cos \left( \frac{x - \lambda t}{2} \right) \right]^2, \quad |x - \lambda t| \leq \pi. \quad (3.2)$$

A particle approximation for Eq. (3.1) can be obtained in the following procedure. First we rewrite (3.1) as  $u_t + (a(x, t)u)_x = 0$ , where

$$a(x, t) = u(x, t) + u_{xx}(x, t). \quad (3.3)$$

We expect the solutions to (3.1) to develop nonsmooth fronts of the form (3.2), and hence, we replace the velocity  $a(x, t)$  in (3.3) with the smoother

$$a_\zeta(x, t) = u * \zeta_\epsilon + u * \zeta'_\epsilon = u_N^\epsilon(x, t) + u_N^\epsilon(x, t)''. \quad (3.4)$$

A particle approximation for a solution of (3.1) is therefore given by

$$u_N^\epsilon(x, t) = \sum_{i=1}^N w_i \zeta_\epsilon(x - x_i(t)), \quad (3.5)$$

where the cutoff function,  $\zeta_\epsilon(x)$ , satisfies (2.6), and the characteristic curves are given by

$$\begin{cases} \frac{dx_i}{dt} = \sum_{j=1}^N w_j \zeta_\epsilon(x_i - x_j) + \sum_{j=1}^N w_j \zeta_\epsilon''(x_i - x_j), \\ x_i(0) = x_i^0. \end{cases} \quad (3.6)$$

*Remarks.* 1. An analogous theorem to Theorem 2.1 for the short time existence and uniqueness of a solution to the dispersion-velocity transport equation (2.8) holds also when  $a_\zeta(x, t)$  is given by (3.4). Even though equation (3.1) is nonlinear the proof of such a theorem is much simpler than the proof of Theorem 2.1, and that is because  $a_\zeta(x, t)$  depends linearly on  $u$  and its derivatives. We skip the details.

2. It was already pointed out in [14] that one cannot expect the delicate balance between the nonlinear advection term and the nonlinear dispersion term (which allows the creation of compactly supported structures) to be preserved on the numerical level.

From that point of view, one of the advantages of our method is that no splitting between the terms is required. One approach in particle methods for approximating solutions to nonlinear problems, such as the Burgers equation or Navier–Stokes equations, is based on a fractional step method, in which the advection part of the equation is solved, followed by a solver to the dissipative part of the equation (see [7]). In the method we present, such a splitting is not required, and that seems to help preserve the properties of the solution.

3. We chose to approximate solutions to (3.1) since this equation enjoys the richness of the features of nonlinear dispersive equations while, from the technical point of view, it is simpler to deal with. (The velocity  $a_\zeta(x, t)$  in its particle approximation depends linearly on  $u$ ). In principle, at least formally, the dispersion-velocity method can be easily extended to other equations as well. For example, a similar method can be written for the  $K(2, 2)$  equation,

$$K(2, 2): u_t + (u^2)_x + (u^2)_{xxx} = 0. \quad (3.7)$$

In this case, the transport velocity is given by

$$a(x, t) = u(x, t) + 2u_{xx}(x, t) + 2\frac{u_x^2(x, t)}{u(x, t)}. \quad (3.8)$$

Another interesting example is a particle approximation for the Korteweg-de Vries equation,

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \quad (3.9)$$

which can be rewritten as  $u_t + (a(x, t)u)_x = 0$  with

$$a(x, t) = u(x, t) + \frac{u_{xx}(x, t)}{u(x, t)}.$$

Since we were mainly interested in this work in studying equations which develop solutions with nonsmooth fronts, we leave the dispersion-velocity approach for the KdV equation for a future study.

#### 4. THE NUMERICAL METHOD

In this section we would like to present the particle method in a general formulation and discuss some of the issues related to its implementation. We therefore consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(u(x, t), x, t)u) = 0, \\ u(x, t = 0) = u_0(x), \end{cases} \quad (4.10)$$

with a velocity  $a(u(x, t), x, t)$  that depends on the problem. For example, the velocity  $a(u(x, t), x, t)$  in the linear equation (2.1) is given by (2.3), while for the nonlinear (3.1) it is given by (3.3).

Given an appropriate smoothing function  $\zeta_\epsilon(x)$ , we can approximate  $a(u(x, t), x, t)$  by

$$a_\zeta(u(x, t), x, t) = a(u(x, t), x, t) * \zeta_\epsilon(x).$$

The *dispersion-velocity transport equation* then takes the form

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a_\zeta u) = 0, \\ u(x, t = 0) = u_0(x). \end{cases} \quad (4.11)$$

The numerical method is obtained by considering a particle approximation as a distribution of the form of

$$u_N^\epsilon(x, t) = \sum_{i=1}^N w_i \zeta_\epsilon(x - x_i(t)), \quad (4.12)$$

where  $x_i(t)$  are the solutions of

$$\begin{cases} \frac{dx_i}{dt} = a_\zeta(u_N^\epsilon, x_i, t), \\ x_i(0) = x_i^0. \end{cases} \quad (4.13)$$

We are now ready to discuss several issues related to the implementation of the method (4.12)–(4.13).

##### 4.1. Initialization

We would like to choose constants  $\{w_i\}$  such that  $u_N(x, 0) = \sum_i w_i \delta(x - x_i(0))$  approximates  $u_0(x)$ . This is done in the sense of measures on  $\mathbf{R}$ .

Given a test function  $\phi \in C_0^0(\mathbf{R})$ , the inner product

$$(u_0(\cdot), \phi(\cdot)) = \int_{\mathbf{R}} u_0(x) \phi(x) dx$$

should be approximated by

$$(u_N(\cdot), \phi(\cdot)) = \sum_i w_i \phi(x_i).$$

In other words, the constants  $\{w_i\}$ , should be determined by solving the standard numerical quadrature problem

$$\int u_0(x) \phi(x) dx \approx \sum_i w_i \phi(x_i). \quad (4.14)$$

One way of solving (4.14) can be, e.g., to cover  $\mathbf{R}$  with a uniform mesh of spacing  $h > 0$ . For  $j \in \mathbf{Z}$  we then denote  $I_j = \{x \mid (j - 1/2)h \leq x \leq (j + 1/2)h\}$ . For example, a midpoint quadrature in  $I_j$  is given by setting

$$w_i = hu_0(x_i).$$

## 4.2. The Cutoff Functions

There is an extensive discussion in the literature on the selection of a cutoff function and its relation to the accuracy of particle methods. At this point we would only like to note that the first cutoff function was introduced by Chorin in [6]. These ideas were further developed in various works, out of which we would like to mention, in particular, the works by Hald [19] and Beale and Majda [2–4]. For a review on the role that cutoff functions play in vortex methods, we refer the reader to Hald [20], the book by Cottet and Koumoutsakos [8], and the review paper by Puckett [32].

For completeness, we would like to present an example for a suitable cutoff function  $\zeta_\epsilon(x)$ . On the real line, a possible  $\zeta_\epsilon(x)$  is a normalized Gaussian

$$\zeta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon^2}}. \quad (4.15)$$

A similar cutoff function can be used in the periodic case if we assume a period  $2L$  which is large enough compared to  $\epsilon$ . In this case, a normalized periodic Gaussian is given by

$$\zeta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \sum_{\tau=-\infty}^{\infty} e^{-\frac{(x-2L\tau)^2}{2\epsilon^2}}, \quad (4.16)$$

or in terms of its Fourier representation, by

$$\zeta_\epsilon(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{1}{2}n^2\epsilon^2\left(\frac{\pi}{L}\right)^2}.$$

## 4.3. Implementation

- We would like to point out that similar to the diffusion-velocity method, the dispersion-velocity method, as formulated in this section, does not allow the solution to change sign. Unlike what happened in the case of the heat equation, the oscillations that the linear dispersive equation generates can cause the solution to change sign. In order to avoid such

undesirable situations, one can add a constant to the initial data so that it stays away from zero, at least for short times.

- There are cases where the velocity  $a_\zeta(u_N^\epsilon, x_i, t)$  has a denominator,  $\mathcal{D}$ , which can vanish (e.g., in the linear problem (2.9)). In order to avoid division by zero, at least from a technical point of view,  $\mathcal{D}^{-1}$  can be replaced by  $\mathcal{D}/(\mathcal{D}^2 + \delta^2)$  with  $\delta$  taken as a small constant, [21].

- It is straightforward to extend the dispersion-velocity method for multidimensional problems. Implementation of particle methods in more than one space dimension is computationally demanding, and there are a lot of methods that were devised in the literature in order to improve the efficiency of the implementation in such cases. We refer the reader to [5, 8, 16, 17, 28, 32] for a review of fast techniques for both particle and vortex methods. We will not deal with efficiency issues in this paper and will leave them for a future publication. A similar comment holds also for resampling issues. From the numerical examples we present below it is clear that when the particles change their location in time, there are situations in which redistributing the particles in space is desirable. Further discussion about redistribution issues can be found in the next section.

## 5. NUMERICAL SIMULATIONS

In this section we present several examples in which we test our new numerical methods for linear as well as for nonlinear problems. For simplicity we used in all of our examples periodic boundary conditions. The time integration was done using a standard, fourth-order Runge–Kutta method with a fixed time step that was chosen small enough to ensure the local stability of the Runge–Kutta method.

- The kernel: In our computations we used two types of smooth kernels. In the linear problems we used the Gaussian kernel given by (4.15). In the nonlinear problems we used the kernel

$$\xi(x) = \frac{1}{\sqrt{\pi}} \left( \frac{3}{2} - x^2 \right) e^{-x^2}. \quad (5.1)$$

This kernel was used in order to reduce the error, even though the overall order of accuracy of the method is observed to be one in both cases. Clearly, the accuracy of the dispersion-velocity method will depend on the choice of the cutoff function  $\zeta_\epsilon(x)$  and on its width  $\epsilon$ . It is possible to improve the order of accuracy of the method by choosing more accurate kernel functions and an optimal choice of the width  $\epsilon$  of the kernel. For an analysis of accuracy of particle methods we refer the reader to [1–3, 11, 30–33].

- Redistribution: Since we are dealing with dispersive equations, we do not expect any bounds on the distance between particles (both lower and upper bounds). In most of the nonlinear problems we tested such a problem was encountered. The technique used to address this issue was a redistribution of the particles in fixed times, which were selected in such a way as to prevent the particles from spreading too far from each other. The new locations and weights of the particles were determined using a third-order spline interpolation. This is not the only possible method, but it did seem to be more accurate than other methods we tried to use (such as redistribution according to (3.5)). It is important to note that local extrema can develop in such high-order reconstructions and therefore, the solution can be expected to change its sign close to zero.

It is well known in particle applications that redistribution of the particles might be crucial for a successful implementation of the method; e.g., see [4, 31]. Without redistribution one might fail to capture the long time behavior of the solution. We encountered such a problem when trying to solve the nonlinear compacton type equations below. In particular, without redistributing the particles, we were not able to pass the stage of the nonlinear interaction between two compactons.

### 5.1. Linear Equations

We start with the linear equation

$$u_t = u_{xxx}, \quad x \in [-\pi, \pi], \quad t \leq 0,$$

subject to initial data  $u(x, 0) = u_0(x)$  and periodic boundary conditions.

First we used the initial data

$$u_0(x) = \cos(x), \quad x \in [-\pi, \pi].$$

In this case the exact solution is a traveling wave  $u(x, t) = \cos(x - t)$ .

The number of particles  $N$  is taken as 40, 80, 160, 320. The width of the Gaussian kernel is taken as  $\epsilon = 0.5\sqrt{h}$ , with  $h = 2\pi/N$ .

A convergence rate study is shown in Table I. The entries in the table are the maximum norm  $\|u - u_N^\epsilon\|_\infty$  and the  $L^2$  norm  $\|u - u_N^\epsilon\|_2$  of the absolute error at a fixed time  $T = 2$ . Also presented are the convergence rate between two grids. The convergence rate is computed as

$$\log_2 \left( \frac{\|u - u_{N_1}^\epsilon\|}{\|u - u_{N_2}^\epsilon\|} \right) / \log_2 \left( \frac{N_2}{N_1} \right), \quad (5.2)$$

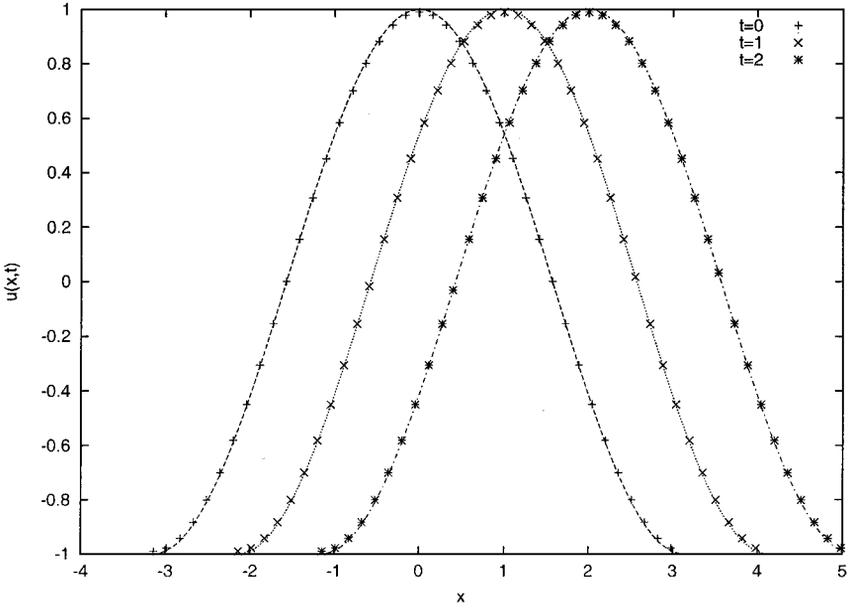
where  $u$  is the projection of the exact solution on the grid,  $u_N^\epsilon$  is the numerical solution, and  $\|u - u_N^\epsilon\|$  is a discrete norm of the absolute error. This table shows a convergence rate which is approximately one. The exact and the approximate solutions of this problem at different times are displayed in Fig. 1.

In the second example, we solved the same equation,  $u_t = u_{xxx}$ , subject to initial data  $u(x, 0) = 5 + \exp(-x^2)$  with periodic boundary conditions on  $[-\pi, \pi]$ . Without the constant in the initial data, the solution would change its sign. The constant does not change

**TABLE I**  
**Convergence Rate for the Linear Problem  $u_t = u_{xxx}$  with Initial Data**  
 **$u(x, 0) = \cos(x)$**

Grid	$\ u - u_N^\epsilon\ _\infty$	$L^\infty$ Convergence rate	$\ u - u_N^\epsilon\ _2$	$L^2$ Convergence rate
$N = 40$	9.8414e-3	—	0.01745	—
$N = 80$	4.8967e-3	1.007	8.6792e-3	1.008
$N = 160$	2.4514e-3	1.003	4.3454e-3	1.002
$N = 320$	1.2424e-3	0.995	2.2021e-3	0.995

Note.  $\epsilon = 0.5\sqrt{h}$ ,  $T = 2$ .



**FIG. 1.** The solution of  $u_t = u_{xxx}$  with initial data  $u_0(x) = \cos(x)$  and periodic boundary conditions on  $[-\pi, \pi]$ .  $N = 40$ ,  $\epsilon = 0.5\sqrt{h}$ . The points represent the location of the particles. The solid lines represent the exact solution.

the solution but it enables us to use the particle method with weights that do not change their sign.

Once again, the cutoff function is taken to be a Gaussian with width  $\epsilon = 0.5\sqrt{h}$ , where  $h = 2\pi/N$  and  $N = 80, 160, 320, 640$ . Since the  $L_2$  norm of the exact solution is preserved, we show in Table II that this feature holds for the numerical solution as well. Figure 2 presents the numerical solution for different times and  $N = 320$ . The points represent the location of the particles at any given time.

### 5.2. Nonlinear Equations

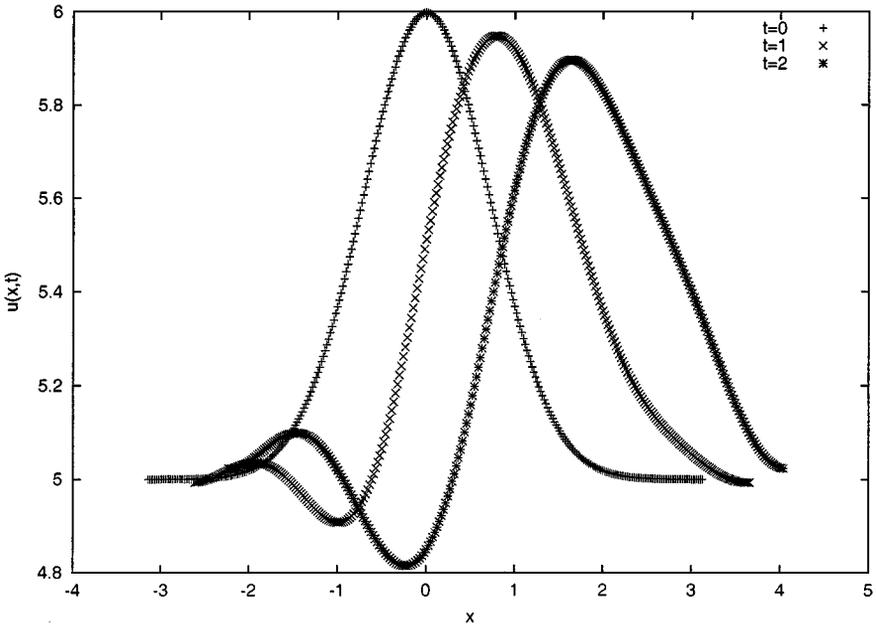
We consider the nonlinear dispersive equation

$$u_t + (u^2)_x + (uu_{xx})_x = 0,$$

**TABLE II**  
**The  $L^2$  Norm of the Solution to the Linear Problem**  
 $u_t = u_{xxx}$  with Initial Data  $u(x, 0) = 5 + e^{-x^2}$

	$T = 0$	$T = 1$	$T = 2$
Grid	$\ u_N^\epsilon\ _2$	$\ u_N^\epsilon\ _2$	$\ u_N^\epsilon\ _2$
$N = 80$	13.26820	13.26827	13.26836
$N = 160$	13.26843	13.26844	13.26847
$N = 320$	13.26854	13.26855	13.26856
$N = 640$	13.26860	13.26859	13.26860

Note.  $\epsilon = 0.5\sqrt{h}$ .



**FIG. 2.** The solution of  $u_t = u_{xxx}$  with initial data  $u_0(x) = 5 + e^{-x^2}$ , and periodic boundary conditions on  $[-\pi, \pi]$ .  $N = 320$ ,  $\epsilon = 0.5\sqrt{h}$ . The points represent the location of the particles.

which generates compacton-type solutions as outlined in Section 3. In all of the examples, the boundary conditions are taken to be periodic in an interval much larger than the compact support of the initial data. The kernel is taken to be in the form (5.1).

### 5.2.1. Compacton Initial Data

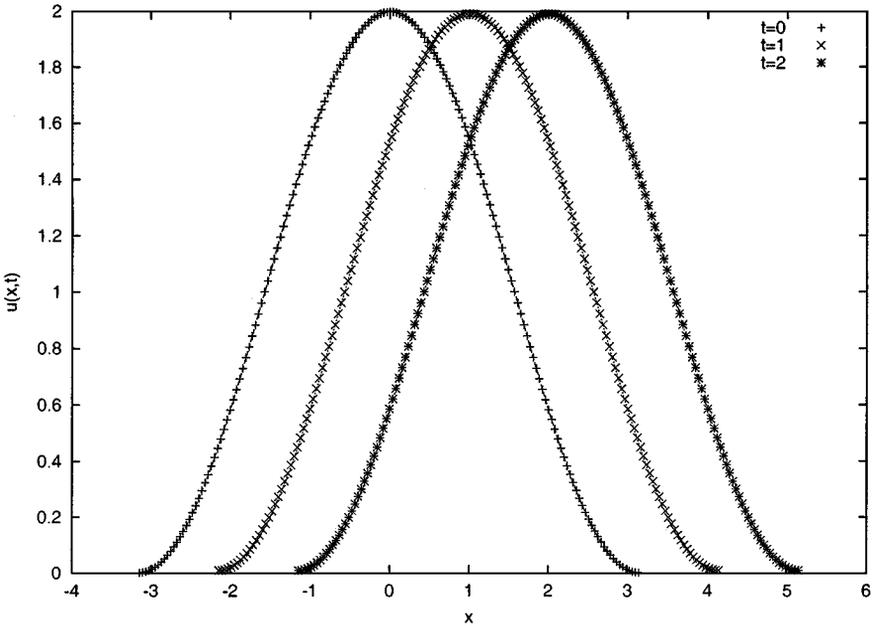
$$u(x, 0) = \begin{cases} 2 \cos^2(x/2), & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}.$$

In this case, the exact solution is a traveling wave given by (3.2) with velocity  $\lambda = 1$ . Figure 3 presents the results of the numerical method for different times, with  $N = 160$  particles taken initially to be equally spaced with spacing  $h$ . The width of the kernel is  $\epsilon = 0.5\sqrt{h}$ . The convergence rate is shown in Table III and is approximately one for both the maximum norm and the  $L^2$  norm.

### 5.2.2. Arbitrary Initial Data

$$u(x, 0) = \begin{cases} 3 \cos^2(x/4), & |x| \leq 2\pi \\ 0, & |x| > 2\pi \end{cases}.$$

In this case we expect the fundamental compactons (3.2) to split out of this initial data. In Fig. 4 we plot the solution in times  $T = 0, 1, 2, 4, 6, 8$ . The number of particles up to time  $T = 1$  was taken as  $N = 200$ . After  $T = 1$ , one hundred additional particles with zero



**FIG. 3.** The solution to (3.1) with initial data  $u_0(x) = 2 \cos^2(x/2)$  on  $[-\pi, \pi]$  and zero elsewhere.  $N = 160$ ,  $\epsilon = 0.5\sqrt{h}$ . The points represent the location of the particles.

weights were added to the right of the solution in order to solve the problem on the entire line. The width of the kernel is taken as  $\epsilon = 1.25\sqrt{h}$ , where  $h$  is the initial spacing between the particles. What can be clearly seen are compactons splitting out of the initial data. In time, the residual tail splits into more compactons (see [34]).

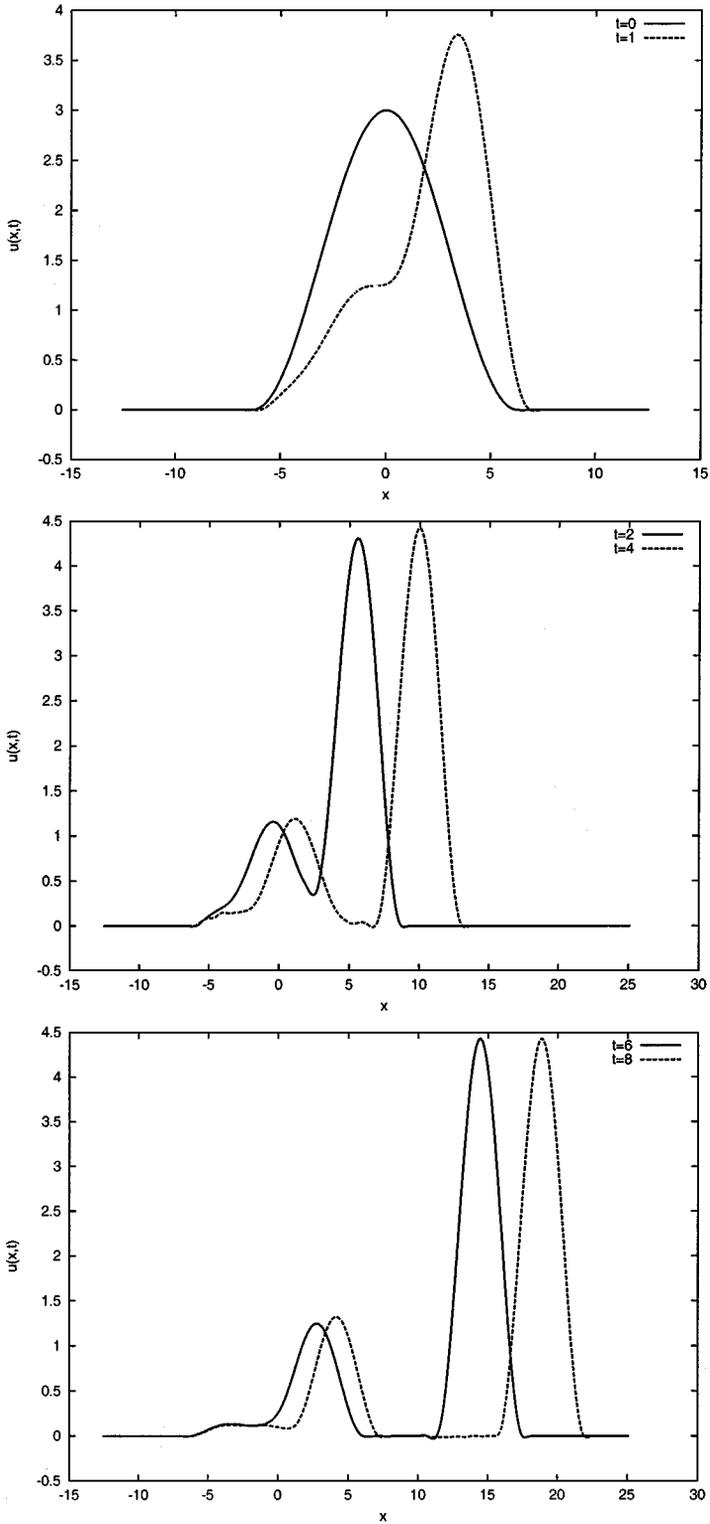
In Fig. 5 we show that the shape of the emerging compactons at time  $T = 8$  coincides with the canonical, fundamental compacton (3.2). The points represent the numerical solution at that time. The solid line represents two fundamental compactons, shifted to the center of the corresponding numerical humps and scaled so as to have the same amplitude.

We also compare our particle method simulations with results that are obtained with a pseudo-spectral method in space and fourth-order Runge–Kutta method in time; see Fig. 6. In order to avoid the numerical oscillations that develop in the pseudo-spectral method from the nonsmooth boundaries we filter the solution every time step with a smooth exponential filter in the Fourier space (for further details see [35]). The number of points in the spectral

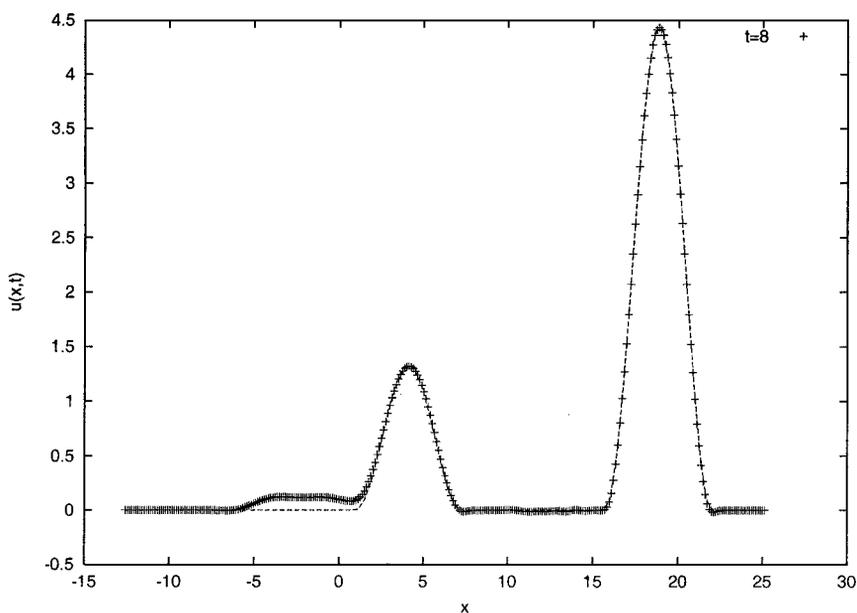
**TABLE III**  
**Convergence Rate for the Solution to (3.1) with Initial Data**  
 $u(x, 0) = 2 \cos^2(x/2)$

Grid	$\ u - u_N^\epsilon\ _\infty$	$L^\infty$ Convergence rate	$\ u - u_N^\epsilon\ _2$	$L^2$ Convergence rate
$N = 40$	0.03851	—	0.06826	—
$N = 80$	0.01945	0.986	0.03446	0.986
$N = 160$	$9.7767e-3$	0.989	0.01732	0.989
$N = 320$	$4.9323e-3$	0.988	$8.7149e-3$	0.989

Note.  $\epsilon = 0.5\sqrt{h}$ ,  $T = 2$ .

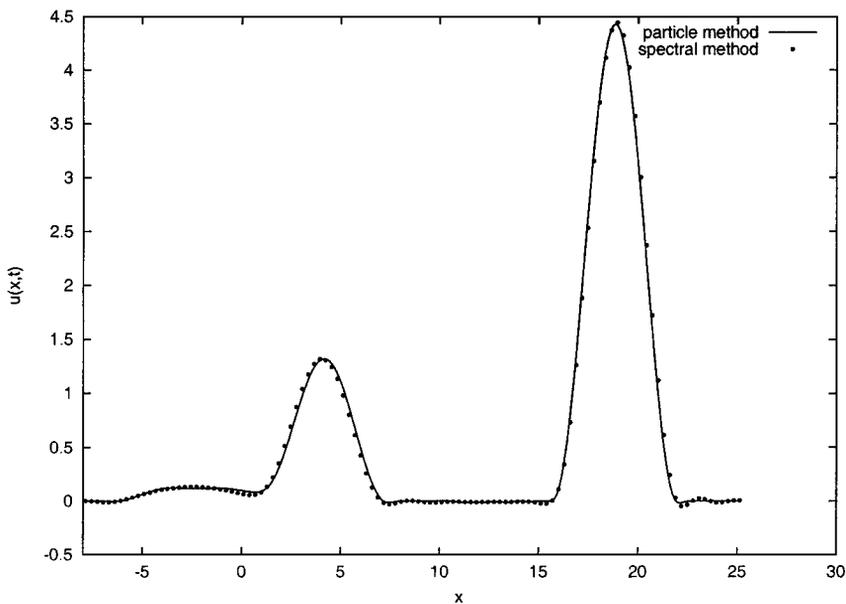


**FIG. 4.** The solution to (3.1) with initial data  $u_0(x) = 3 \cos^2(x/4)$  on  $[-2\pi, 2\pi]$  and zero elsewhere.  $N = 200$  for  $t \leq 1$ ,  $N = 300$  for  $t > 1$ ,  $\epsilon = 1.25\sqrt{h}$ .

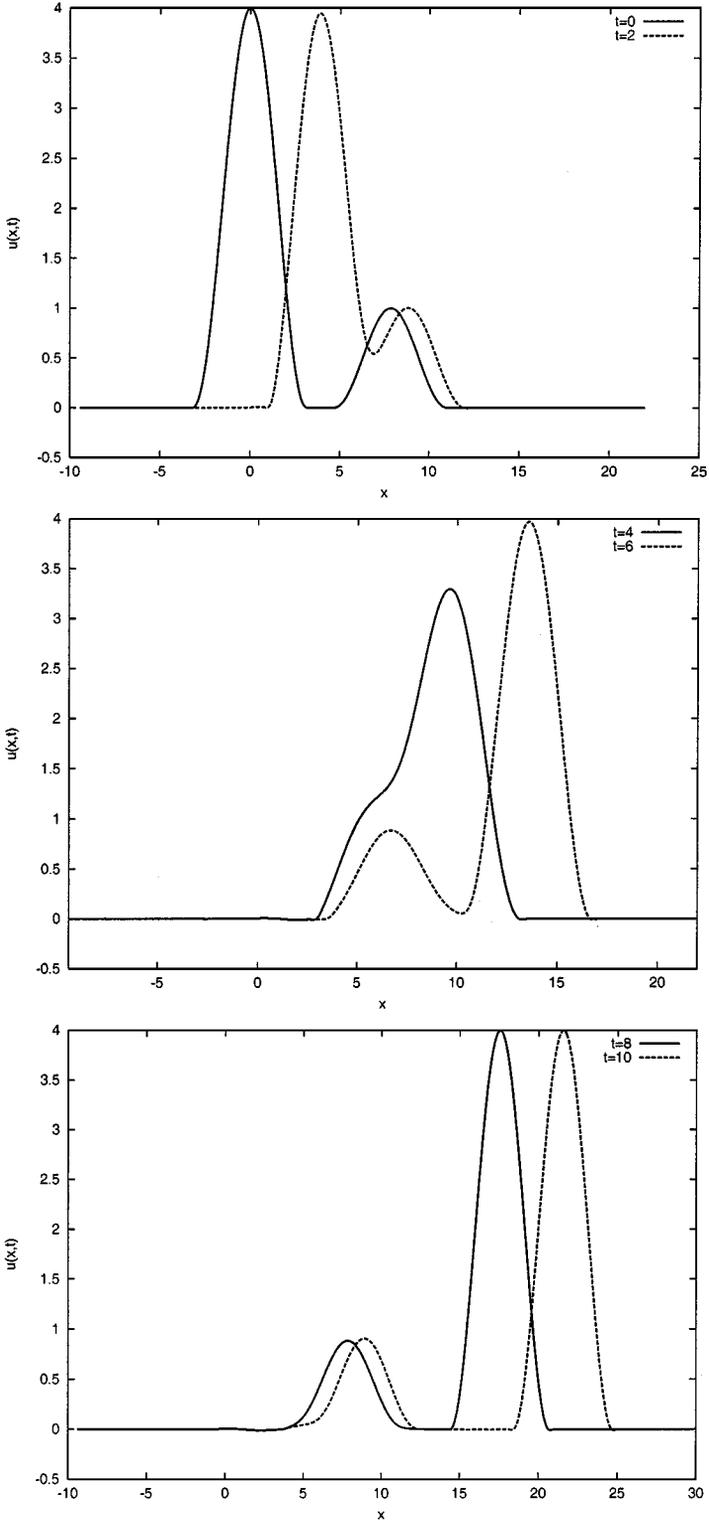


**FIG. 5.** The structures splitting from the initial data are fundamental compactons.  $T = 8$ . The points represent the particle method. The dashed line represents the fundamental compactons (3.2).

simulations is taken as  $N = 128$ . Clearly, the results of the particle method do not suffer from the spurious oscillations that are present in the spectral methods. It is important to note, however, that the similarity between the results obtained with the two methods strengthens also the validity of the spectral methods as a tool for solving problems of this type.



**FIG. 6.** The solution to (3.1) with initial data  $u_0(x) = 3 \cos^2(x/4)$  on  $[-2\pi, 2\pi]$  and zero elsewhere.  $T = 8$ . The points represent the spectral method. The solid line represents the particle method.



**FIG. 7.** The solution to (3.1) with initial data (5.2.3).  $N = 400$  for  $t \leq 4$ ,  $N = 500$  for  $t > 4$ ,  $\epsilon = \sqrt{h}$ .

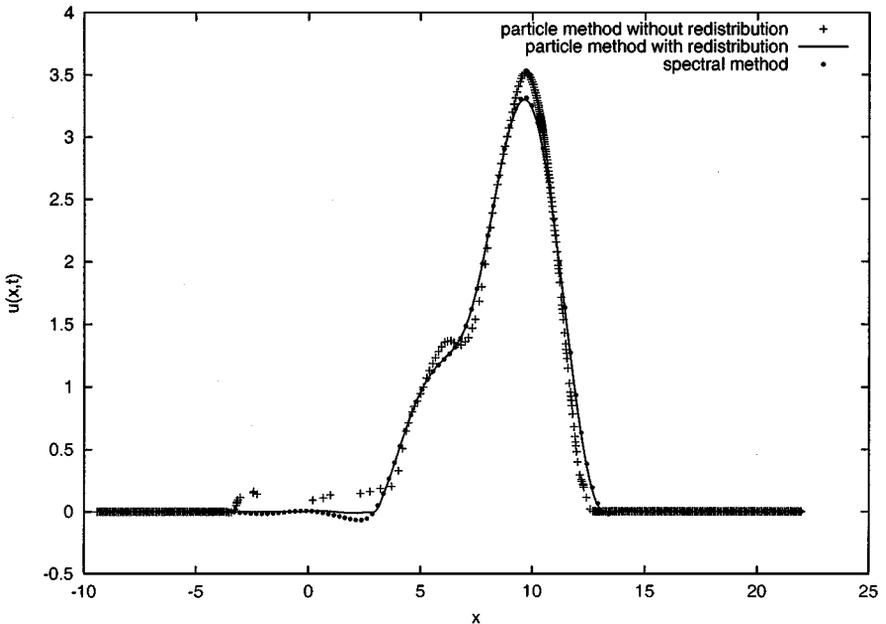
### 5.2.3. Compacton–Compacton Interaction

Here the initial condition is taken as two compactons:

$$u(x, 0) = \begin{cases} 4 \cos^2(x/2), & -\pi < x < \pi \\ \cos^2((x - 2.5\pi)/2), & 1.5\pi < x < 3.5\pi \\ 0, & \text{elsewhere.} \end{cases}$$

In Fig. 7 we plot the solution in times  $T = 0, 2, 4, 6, 8, 10$ . Up to time  $T = 4$  the number of particles was taken as  $N = 400$ . After  $T = 4$ , one hundred additional particles with zero weights were added to the right of the solution in order to solve the problem on the entire line. The width of the kernel is taken as  $\epsilon = \sqrt{h}$ , where  $h$  is the initial spacing between the particles. The higher compacton (to the left) that travels with a higher velocity ( $\lambda = 2$ ) passes through the lower compacton which travels slower ( $\lambda = 0.5$ ) after going through a nonlinear interaction that generates a phase shift. Evidently, the particles are capable of capturing the nonlinear interaction. We would like to note that the compactons seem to emerge from the interaction in the canonical compacton shape (3.2) while leaving behind a small residue. A similar phenomenon was observed in the past when approximating solutions to related equations with other methods; see for example [34, 35].

Finally, in Fig. 8 we compare the solutions obtained at time  $T = 4$  by both the particle and the pseudo-spectral method outlined above. The spurious numerical oscillations that were presented in the spectral computation (even though the solution is filtered in every time step) completely disappear in the particle computation. In this figure we also show the results obtained when the particle method is run without any redistribution of the particles.



**FIG. 8.** The solution to (3.1) with initial data (5.2.3).  $T = 4$ . The plus represents the particle method without any redistribution of particles. The solid line represents the particle method with redistribution. The dot represents the spectral methods.

Redistribution is therefore essential; without it the compacton–compacton interaction cannot be captured. Redistribution was applied in fixed time intervals of  $\Delta t = 0.25$  using third-order splines as described in the beginning of the section.

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